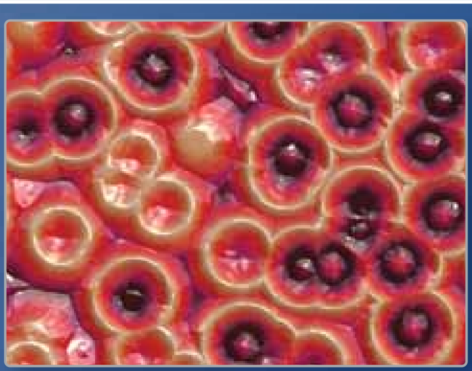
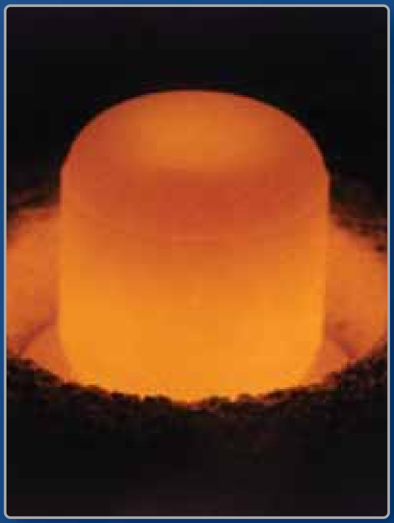
6 Exponential Functions and Sequences

Fibonacci and Flowers *(p. 343)*

Soup Kitchen *(p. 338)*



Bacterial Culture *(p. 330)*



Plutonium Decay *(p. 321)*

Coyote Population *(p. 311)*

Maintaining Mathematical Proficiency

Using Order of Operations

Example 1 **Evaluate $10^2 \div (30 \div 3) - 4(3 - 9) + 5^1$.**

First:	Parentheses	$10^2 \div (30 \div 3) - 4(3 - 9) + 5^1 = 10^2 \div 10 - 4(-6) + 5^1$
Second:	Exponents	$= 100 \div 10 - 4(-6) + 5$
Third:	Multiplication and Division (from left to right)	$= 10 + 24 + 5$
Fourth:	Addition and Subtraction (from left to right)	$= 39$

Evaluate the expression.

1. $12\left(\frac{14}{2}\right) - 3^3 + 15 - 9^2$ **2.** $5^2 \cdot 8 \div 2^2 + 20 \cdot 3 - 4$ **3.** $-7 + 16 \div 2^4 + (10 - 4^2)$

Finding Square Roots

Example 2 **Find $-\sqrt{81}$.**

▶ $-\sqrt{81}$ represents the negative square root. Because $9^2 = 81$, $-\sqrt{81} = -\sqrt{9^2} = -9$.

Find the square root(s).

4. $\sqrt{64}$ **5.** $-\sqrt{4}$ **6.** $-\sqrt{25}$ **7.** $\pm\sqrt{121}$

Writing Equations for Arithmetic Sequences

Example 3 **Write an equation for the *n*th term of the arithmetic sequence 5, 15, 25, 35, ...**

The first term is 5, and the common difference is 10.

$a_n = a_1 + (n - 1)d$	Equation for an arithmetic sequence
$a_n = 5 + (n - 1)(10)$	Substitute 5 for a_1 and 10 for d.
$a_n = 10n - 5$	Simplify.

Write an equation for the *n*th term of the arithmetic sequence.

8. 12, 14, 16, 18, . . . **9.** 6, 3, 0, −3, . . . **10.** 22, 15, 8, 1, . . .

11. **ABSTRACT REASONING** Recall that a perfect square is a number with integers as its square roots. Is the product of two perfect squares always a perfect square? Is the quotient of two perfect squares always a perfect square? Explain your reasoning.

Mathematical Practices

Mathematically proficient students look closely to find a pattern.

Problem-Solving Strategies

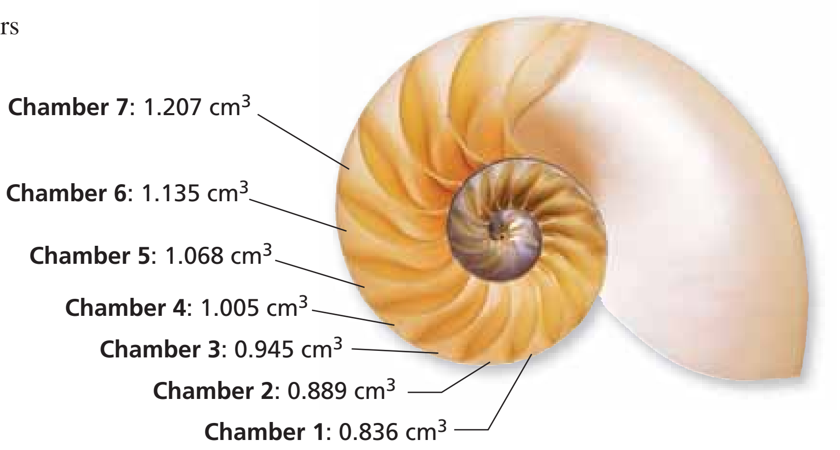
Core Concept

Finding a Pattern

When solving a real-life problem, look for a pattern in the data. The pattern could include repeating items, numbers, or events. After you find the pattern, describe it and use it to solve the problem.

EXAMPLE 1 Using a Problem-Solving Strategy

The volumes of seven chambers of a chambered nautilus are given. Find the volume of Chamber 10.



SOLUTION

To find a pattern, try dividing each volume by the volume of the previous chamber.

$$\frac{0.889}{0.836} \approx 1.063 \qquad \frac{0.945}{0.889} \approx 1.063 \qquad \frac{1.005}{0.945} \approx 1.063$$

$$\frac{1.068}{1.005} \approx 1.063 \qquad \frac{1.135}{1.068} \approx 1.063 \qquad \frac{1.207}{1.135} \approx 1.063$$

From this, you can see that the volume of each chamber is about 6.3% greater than the volume of the previous chamber. To find the volume of Chamber 10, multiply the volume of Chamber 7 by 1.063 three times.

$$1.207(1.063) \approx \underset{\substack{\uparrow \\ \text{volume of Chamber 8}}}{\mathbf{1.283}} \qquad 1.283(1.063) \approx \underset{\substack{\uparrow \\ \text{volume of Chamber 9}}}{\mathbf{1.364}} \qquad 1.364(1.063) \approx \underset{\substack{\uparrow \\ \text{volume of Chamber 10}}}{\mathbf{1.450}}$$

▶ The volume of Chamber 10 is about 1.450 cubic centimeters.

Monitoring Progress

1. A rabbit population over 8 consecutive years is given by 50, 80, 128, 205, 328, 524, 839, 1342. Find the population in the tenth year.
2. The sums of the numbers in the first eight rows of Pascal's Triangle are 1, 2, 4, 8, 16, 32, 64, 128. Find the sum of the numbers in the tenth row.

6.1 Properties of Exponents

Essential Question How can you write general rules involving properties of exponents?

EXPLORATION 1 Writing Rules for Properties of Exponents

Work with a partner.

WRITING GENERAL RULES

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results in writing general rules.

a. What happens when you multiply two powers with the same base? Write the product of the two powers as a single power. Then write a *general rule* for finding the product of two powers with the same base.

i. $(2^2)(2^3) =$ **ii.** $(4^1)(4^5) =$

iii. $(5^3)(5^5) =$ **iv.** $(x^2)(x^6) =$

b. What happens when you divide two powers with the same base? Write the quotient of the two powers as a single power. Then write a *general rule* for finding the quotient of two powers with the same base.

i. $\frac{4^3}{4^2} =$ **ii.** $\frac{2^5}{2^2} =$

iii. $\frac{x^6}{x^3} =$ **iv.** $\frac{3^4}{3^4} =$

c. What happens when you find a power of a power? Write the expression as a single power. Then write a *general rule* for finding a power of a power.

i. $(2^2)^4 =$ **ii.** $(7^3)^2 =$

iii. $(y^3)^3 =$ **iv.** $(x^4)^2 =$

d. What happens when you find a power of a product? Write the expression as the product of two powers. Then write a *general rule* for finding a power of a product.

i. $(2 \cdot 5)^2 =$ **ii.** $(5 \cdot 4)^3 =$

iii. $(6a)^2 =$ **iv.** $(3x)^2 =$

e. What happens when you find a power of a quotient? Write the expression as the quotient of two powers. Then write a *general rule* for finding a power of a quotient.

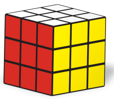
i. $\left(\frac{2}{3}\right)^2 =$ **ii.** $\left(\frac{4}{3}\right)^3 =$

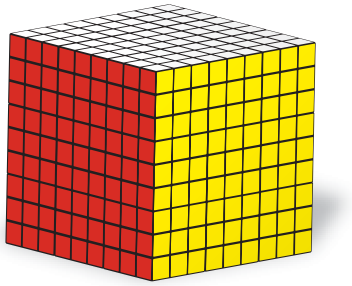
iii. $\left(\frac{x}{2}\right)^3 =$ **iv.** $\left(\frac{a}{b}\right)^4 =$

Communicate Your Answer

2. How can you write general rules involving properties of exponents?

3. There are 3^3 small cubes in the cube below. Write an expression for the number of small cubes in the large cube at the right.





6.1 Lesson

Core Vocabulary

Previous
power
exponent
base
scientific notation

What You Will Learn

- Use zero and negative exponents.
- Use the properties of exponents.
- Solve real-life problems involving exponents.

Using Zero and Negative Exponents

Core Concept

Zero Exponent

Words For any nonzero number a, $a^0 = 1$. The power 0^0 is undefined.

Numbers $4^0 = 1$ **Algebra** $a^0 = 1$, where $a \neq 0$

Negative Exponents

Words For any integer n and any nonzero number a, a^{-n} is the reciprocal of a^n.

Numbers $4^{-2} = \frac{1}{4^2}$ **Algebra** $a^{-n} = \frac{1}{a^n}$, where $a \neq 0$

EXAMPLE 1 Using Zero and Negative Exponents

Evaluate each expression.

a. 6.7^0 **b.** $(-2)^{-4}$

SOLUTION

a. $6.7^0 = 1$ Definition of zero exponent

b. $(-2)^{-4} = \frac{1}{(-2)^4}$ Definition of negative exponent

$= \frac{1}{16}$ Simplify.

EXAMPLE 2 Simplifying an Expression

Simplify the expression $\frac{4x^0}{y^{-3}}$. Write your answer using only positive exponents.

SOLUTION

$\frac{4x^0}{y^{-3}} = 4x^0y^3$ Definition of negative exponent

$= 4y^3$ Definition of zero exponent

Monitoring Progress

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Evaluate the expression.

1. $(-9)^0$ **2.** 3^{-3} **3.** $\frac{-5^0}{2^{-2}}$

4. Simplify the expression $\frac{3^{-2}x^{-5}}{y^0}$. Write your answer using only positive exponents.

Using the Properties of Exponents

REMEMBER

The expression x^3 is called a *power*. The *base*, x, is used as a factor 3 times because the *exponent* is 3.

Core Concept

Product of Powers Property

Let a be a real number, and let m and n be integers.

Words To multiply powers with the same base, add their exponents.

Numbers $4^6 \cdot 4^3 = 4^{6+3} = 4^9$ **Algebra** $a^m \cdot a^n = a^{m+n}$

Quotient of Powers Property

Let a be a nonzero real number, and let m and n be integers.

Words To divide powers with the same base, subtract their exponents.

Numbers $\frac{4^6}{4^3} = 4^{6-3} = 4^3$ **Algebra** $\frac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$

Power of a Power Property

Let a be a real number, and let m and n be integers.

Words To find a power of a power, multiply the exponents.

Numbers $(4^6)^3 = 4^{6 \cdot 3} = 4^{18}$ **Algebra** $(a^m)^n = a^{mn}$

EXAMPLE 3 Using Properties of Exponents

Simplify each expression. Write your answer using only positive exponents.

a. $3^2 \cdot 3^6$ **b.** $\frac{(-4)^2}{(-4)^7}$ **c.** $(z^4)^{-3}$

SOLUTION

a. $3^2 \cdot 3^6 = 3^{2+6}$ Product of Powers Property

$= 3^8 = 6561$ Simplify.

b. $\frac{(-4)^2}{(-4)^7} = (-4)^{2-7}$ Quotient of Powers Property

$= (-4)^{-5}$ Simplify.

$= \frac{1}{(-4)^5} = -\frac{1}{1024}$ Definition of negative exponent

c. $(z^4)^{-3} = z^{4 \cdot (-3)}$ Power of a Power Property

$= z^{-12}$ Simplify.

$= \frac{1}{z^{12}}$ Definition of negative exponent

Monitoring Progress

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Simplify the expression. Write your answer using only positive exponents.

5. $10^4 \cdot 10^{-6}$ **6.** $x^9 \cdot x^{-9}$ **7.** $\frac{-5^8}{-5^4}$

8. $\frac{y^6}{y^7}$ **9.** $(6^{-2})^{-1}$ **10.** $(w^{12})^5$

Core Concept

Power of a Product Property

Let a and b be real numbers, and let m be an integer.

Words To find a power of a product, find the power of each factor and multiply.

Numbers $(3 \cdot 2)^5 = 3^5 \cdot 2^5$ **Algebra** $(ab)^m = a^m b^m$

Power of a Quotient Property

Let a and b be real numbers with $b \neq 0$, and let m be an integer.

Words To find the power of a quotient, find the power of the numerator and the power of the denominator and divide.

Numbers $\left(\frac{3}{2}\right)^5 = \frac{3^5}{2^5}$ **Algebra** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, where $b \neq 0$

EXAMPLE 4 Using Properties of Exponents

Simplify each expression. Write your answer using only positive exponents.

a. $(-1.5y)^2$ **b.** $\left(\frac{a}{-10}\right)^3$ **c.** $\left(\frac{3d}{2}\right)^4$ **d.** $\left(\frac{2x}{3}\right)^{-5}$

SOLUTION

a. $(-1.5y)^2 = (-1.5)^2 \cdot y^2$ Power of a Product Property

$= 2.25y^2$ Simplify.

b. $\left(\frac{a}{-10}\right)^3 = \frac{a^3}{(-10)^3}$ Power of a Quotient Property

$= -\frac{a^3}{1000}$ Simplify.

c. $\left(\frac{3d}{2}\right)^4 = \frac{(3d)^4}{2^4}$ Power of a Quotient Property

$= \frac{3^4d^4}{2^4}$ Power of a Product Property

$= \frac{81d^4}{16}$ Simplify.

d. $\left(\frac{2x}{3}\right)^{-5} = \frac{(2x)^{-5}}{3^{-5}}$ Power of a Quotient Property

$= \frac{3^5}{(2x)^5}$ Definition of negative exponent

$= \frac{3^5}{2^5x^5}$ Power of a Product Property

$= \frac{243}{32x^5}$ Simplify.

ANOTHER WAY

Because the exponent is negative, you could find the reciprocal of the base first. Then simplify.

$\left(\frac{2x}{3}\right)^{-5} = \left(\frac{3}{2x}\right)^5 = \frac{243}{32x^5}$

Monitoring Progress

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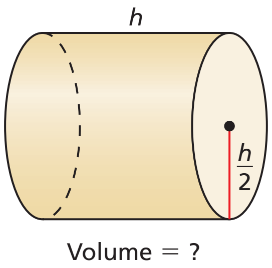
Simplify the expression. Write your answer using only positive exponents.

11. $(10y)^{-3}$ **12.** $\left(-\frac{4}{n}\right)^5$ **13.** $\left(\frac{1}{2k^2}\right)^5$ **14.** $\left(\frac{6c}{7}\right)^{-2}$

Solving Real-Life Problems

EXAMPLE 5 Simplifying a Real-Life Expression

Which of the expressions shown represent the volume of the cylinder, where r is the radius and h is the height?



$2\pi r^3$	$\pi h^3 2^{-2}$	$\pi h 4^{-1}$
$\frac{\pi h^2}{4}$	$\frac{\pi h^3}{4}$	$\frac{\pi h^3}{2}$

SOLUTION

$$V = \pi r^2 h$$ Formula for the volume of a cylinder

$$= \pi\left(\frac{h}{2}\right)^2(h)$$ Substitute $\frac{h}{2}$ for r.

$$= \pi\left(\frac{h^2}{2^2}\right)(h)$$ Power of a Quotient Property

$$= \frac{\pi h^3}{4}$$ Simplify.

Any expression equivalent to $\frac{\pi h^3}{4}$ represents the volume of the cylinder.

- You can use the properties of exponents to write $\pi h^3 2^{-2}$ as $\frac{\pi h^3}{4}$.
- Note $h = 2r$. When you substitute $2r$ for h in $\frac{\pi h^3}{4}$, you can write $\frac{\pi(2r)^3}{4}$ as $2\pi r^3$.
- None of the other expressions are equivalent to $\frac{\pi h^3}{4}$.

▶ The expressions $2\pi r^3$, $\pi h^3 2^{-2}$, and $\frac{\pi h^3}{4}$ represent the volume of the cylinder.

REMEMBER

A number is written in scientific notation when it is of the form $a \times 10^b$, where $1 \le a < 10$ and b is an integer.

EXAMPLE 6 Solving a Real-Life Problem

A jellyfish emits about 1.25×10^8 particles of light, or photons, in 6.25×10^{-4} second. How many photons does the jellyfish emit each second? Write your answer in scientific notation and in standard form.

SOLUTION

Divide to find the unit rate.

$$\frac{1.25 \times 10^8}{6.25 \times 10^{-4}} \begin{matrix} \leftarrow \text{photons} \\ \leftarrow \text{seconds} \end{matrix}$$ Write the rate.

$$= \frac{1.25}{6.25} \times \frac{10^8}{10^{-4}}$$ Rewrite.

$$= 0.2 \times 10^{12}$$ Simplify.

$$= 2 \times 10^{11}$$ Write in scientific notation.

▶ The jellyfish emits 2×10^{11}, or 200,000,000,000 photons per second.

Monitoring Progress

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15. Write two expressions that represent the area of a base of the cylinder in Example 5.

16. It takes the Sun about 2.3×10^8 years to orbit the center of the Milky Way. It takes Pluto about 2.5×10^2 years to orbit the Sun. How many times does Pluto orbit the Sun while the Sun completes one orbit around the center of the Milky Way? Write your answer in scientific notation.

6.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Which definitions or properties would you use to simplify the expression $(4^8 \cdot 4^{-4})^{-2}$? Explain.

2. **WRITING** Explain when and how to use the Power of a Product Property.

3. **WRITING** Explain when and how to use the Quotient of Powers Property.

4. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

Simplify $3^3 \cdot 3^6$.	Simplify 3^{3+6}.	Simplify $3^{6 \cdot 3}$.	Simplify $3^6 \cdot 3^3$.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, evaluate the expression. *(See Example 1.)*

5. $(-7)^0$

6. 4^0

7. 5^{-4}

8. $(-2)^{-5}$

9. $\dfrac{2^{-4}}{4^0}$

10. $\dfrac{5^{-1}}{-9^0}$

11. $\dfrac{-3^{-3}}{6^{-2}}$

12. $\dfrac{(-8)^{-2}}{3^{-4}}$

In Exercises 13–22, simplify the expression. Write your answer using only positive exponents. *(See Example 2.)*

13. x^{-7}

14. y^0

15. $9x^0y^{-3}$

16. $15c^{-8}d^0$

17. $\dfrac{2^{-2}m^{-3}}{n^0}$

18. $\dfrac{10^0r^{-11}s}{3^2}$

19. $\dfrac{4^{-3}a^0}{b^{-7}}$

20. $\dfrac{p^{-8}}{7^{-2}q^{-9}}$

21. $\dfrac{2^2y^{-6}}{8^{-1}z^0x^{-7}}$

22. $\dfrac{13x^{-5}y^0}{5^{-3}z^{-10}}$

In Exercises 23–32, simplify the expression. Write your answer using only positive exponents. *(See Example 3.)*

23. $\dfrac{5^6}{5^2}$

24. $\dfrac{(-6)^8}{(-6)^5}$

25. $(-9)^2 \cdot (-9)^2$

26. $4^{-5} \cdot 4^5$

27. $(p^6)^4$

28. $(s^{-5})^3$

29. $6^{-8} \cdot 6^5$

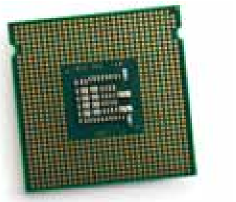
30. $-7 \cdot (-7)^{-4}$

31. $\dfrac{x^5}{x^4} \cdot x$

32. $\dfrac{z^8 \cdot z^2}{z^5}$

33. USING PROPERTIES A microscope magnifies an object 10^5 times. The length of an object is 10^{-7} meter. What is its magnified length?

34. USING PROPERTIES The area of the rectangular computer chip is $112a^3b^2$ square microns. What is the length?



ERROR ANALYSIS In Exercises 35 and 36, describe and correct the error in simplifying the expression.

35.

✗

$$\begin{aligned} 2^4 \cdot 2^5 &= (2 \cdot 2)^{4+5} \\ &= 4^9 \end{aligned}$$

36.

✗

$$\begin{aligned} \frac{x^5 \cdot x^3}{x^4} &= \frac{x^8}{x^4} \\ &= x^{8/4} \\ &= x^2 \end{aligned}$$

In Exercises 37–44, simplify the expression. Write your answer using only positive exponents. *(See Example 4.)*

37. $(-5z)^3$

38. $(4x)^{-4}$

39. $\left(\frac{6}{n}\right)^{-2}$

40. $\left(\frac{-t}{3}\right)^{2}$

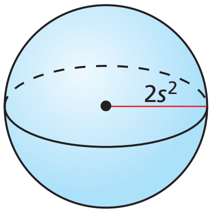
41. $(3s^8)^{-5}$

42. $(-5p^3)^3$

43. $\left(-\frac{w^3}{6}\right)^{-2}$

44. $\left(\frac{1}{2r^6}\right)^{-6}$

45. USING PROPERTIES Which of the expressions represent the volume of the sphere? Explain. *(See Example 5.)*



Ⓐ $\left(\frac{3s^2}{2^4\pi s^8}\right)^{-1}$

Ⓑ $(2^5\pi s^6)(3^{-1})$

Ⓒ $\frac{32\pi s^6}{3}$

Ⓓ $(2s)^5 \bullet \frac{\pi s}{3}$

Ⓔ $\left(\frac{3\pi s^6}{32}\right)^{-1}$

Ⓕ $\frac{32}{3}\pi s^5$

46. MODELING WITH MATHEMATICS Diffusion is the movement of molecules from one location to another. The time t (in seconds) it takes molecules to diffuse a distance of x centimeters is given by $t = \frac{x^2}{2D}$, where D is the diffusion coefficient. The diffusion coefficient for a drop of ink in water is about 10^{-5} square centimeters per second. How long will it take the ink to diffuse 1 micrometer (10^{-4} centimeter)?

In Exercises 47–50, simplify the expression. Write your answer using only positive exponents.

47. $\left(\frac{2x^{-2}y^3}{3xy^{-4}}\right)^4$

48. $\left(\frac{4s^5t^{-7}}{-2s^{-2}t^4}\right)^3$

49. $\left(\frac{3m^{-5}n^2}{4m^{-2}n^0}\right)^2 \bullet \left(\frac{mn^4}{9n}\right)^2$

50. $\left(\frac{3x^3y^0}{x^{-2}}\right)^4 \bullet \left(\frac{y^2x^{-4}}{5xy^{-8}}\right)^3$

In Exercises 51–54, evaluate the expression. Write your answer in scientific notation and standard form.

51. $(3 \times 10^2)(1.5 \times 10^{-5})$

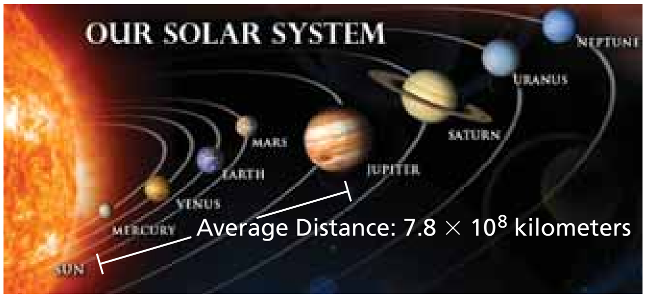
52. $(6.1 \times 10^{-3})(8 \times 10^9)$

53. $\frac{(6.4 \times 10^7)}{(1.6 \times 10^5)}$

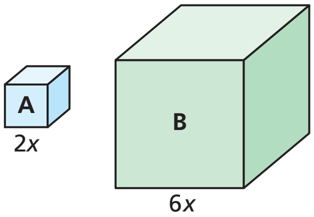
54. $\frac{(3.9 \times 10^{-5})}{(7.8 \times 10^{-8})}$

55. PROBLEM SOLVING In 2012, on average, about 9.46×10^{-1} pound of potatoes was produced for every 2.3×10^{-5} acre harvested. How many pounds of potatoes on average were produced for each acre harvested? Write your answer in scientific notation and in standard form. *(See Example 6.)*

56. PROBLEM SOLVING The speed of light is approximately 3×10^5 kilometers per second. How long does it take sunlight to reach Jupiter? Write your answer in scientific notation and in standard form.



57. MATHEMATICAL CONNECTIONS Consider Cube A and Cube B.



a. Which property of exponents should you use to simplify an expression for the volume of each cube?

b. How can you use the Power of a Quotient Property to find how many times greater the volume of Cube B is than the volume of Cube A?

58. PROBLEM SOLVING A byte is a unit used to measure a computer's memory. The table shows the numbers of bytes in several units of measure.

Unit	kilobyte	megabyte	gigabyte	terabyte
Number of bytes	2^{10}	2^{20}	2^{30}	2^{40}

a. How many kilobytes are in 1 terabyte?

b. How many megabytes are in 16 gigabytes?

c. Another unit used to measure a computer's memory is a bit. There are 8 bits in a byte. How can you convert the number of bytes in each unit of measure given in the table to bits? Can you still use a base of 2? Explain.

REWRITING EXPRESSIONS **In Exercises 59–62, rewrite the expression as a power of a product.**

59. $8a^3b^3$

60. $16r^2s^2$

61. $64w^{18}z^{12}$

62. $81x^4y^8$

63. **USING STRUCTURE** The probability of rolling a 6 on a number cube is $\frac{1}{6}$. The probability of rolling a 6 twice in a row is $\left(\frac{1}{6}\right)^2 = \frac{1}{36}$.

a. Write an expression that represents the probability of rolling a 6 n times in a row.

b. What is the probability of rolling a 6 four times in a row?

c. What is the probability of flipping heads on a coin five times in a row? Explain.

64. **HOW DO YOU SEE IT?** The shaded part of Figure n represents the portion of a piece of paper visible after folding the paper in half n times.

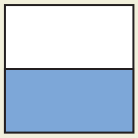


Figure 1

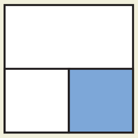


Figure 2

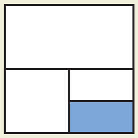


Figure 3

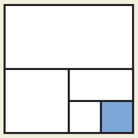


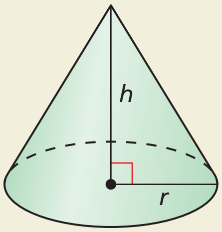
Figure 4

a. What fraction of the original piece of paper is each shaded part?

b. Rewrite each fraction from part (a) in the form 2^x.

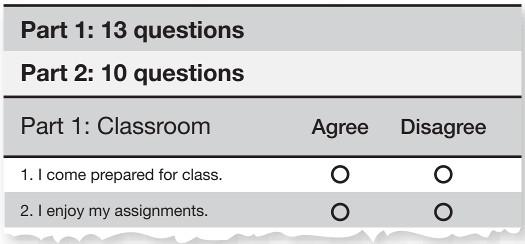
65. **REASONING** Find x and y when $\frac{b^x}{b^y} = b^9$ and $\frac{b^x \cdot b^2}{b^{3y}} = b^{13}$. Explain how you found your answer.

66. **THOUGHT PROVOKING** Write expressions for r and h so that the volume of the cone can be represented by the expression $27\pi x^8$. Find r and h.



67. **MAKING AN ARGUMENT** One of the smallest plant seeds comes from an orchid, and one of the largest plant seeds comes from a double coconut palm. A seed from an orchid has a mass of 10^{-6} gram. The mass of a seed from a double coconut palm is 10^{10} times the mass of the seed from the orchid. Your friend says that the seed from the double coconut palm has a mass of about 1 kilogram. Is your friend correct? Explain.

68. **CRITICAL THINKING** Your school is conducting a survey. Students can answer the questions in either part with "agree" or "disagree."



Part 1: 13 questions

Part 2: 10 questions

Part 1: Classroom	Agree	Disagree
1. I come prepared for class.	○	○
2. I enjoy my assignments.	○	○

a. What power of 2 represents the number of different ways that a student can answer all the questions in Part 1?

b. What power of 2 represents the number of different ways that a student can answer all the questions on the entire survey?

c. The survey changes, and students can now answer "agree," "disagree," or "no opinion." How does this affect your answers in parts (a) and (b)?

69. **ABSTRACT REASONING** Compare the values of a^n and a^{-n} when $n < 0$, when $n = 0$, and when $n > 0$ for (a) $a > 1$ and (b) $0 < a < 1$. Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the square root(s). *(Skills Review Handbook)*

70. $\sqrt{25}$

71. $-\sqrt{100}$

72. $\pm\sqrt{\frac{1}{64}}$

Classify the real number in as many ways as possible. *(Skills Review Handbook)*

73. 12

74. $\frac{65}{9}$

75. $\frac{\pi}{4}$

6.2 Radicals and Rational Exponents

Essential Question

How can you write and evaluate an *n*th root of a number?

Recall that you cube a number as follows.

3rd power

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$ 2 cubed is 8.

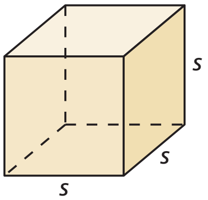
To "undo" cubing a number, take the cube root of the number.

Symbol for cube root is $\sqrt[3]{}$.

$$\sqrt[3]{8} = \sqrt[3]{2^3} = 2$$ The cube root of 8 is 2.

EXPLORATION 1 Finding Cube Roots

Work with a partner. Use a cube root symbol to write the side length of each cube. Then find the cube root. Check your answers by multiplying. Which cube is the largest? Which two cubes are the same size? Explain your reasoning.



a. Volume $= 27\text{ ft}^3$

b. Volume $= 125\text{ cm}^3$

c. Volume $= 3375\text{ in.}^3$

d. Volume $= 3.375\text{ m}^3$

e. Volume $= 1\text{ yd}^3$

f. Volume $= \frac{125}{8}\text{ mm}^3$

JUSTIFYING CONCLUSIONS

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2 Estimating *n*th Roots

Work with a partner. Estimate each positive *n*th root. Then match each *n*th root with the point on the number line. Justify your answers.

a. $\sqrt[4]{25}$

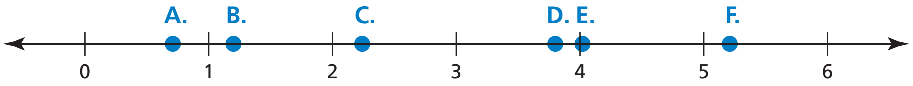
b. $\sqrt{0.5}$

c. $\sqrt[5]{2.5}$

d. $\sqrt[3]{65}$

e. $\sqrt[3]{55}$

f. $\sqrt[6]{20{,}000}$



Communicate Your Answer

3. How can you write and evaluate an *n*th root of a number?

4. The body mass *m* (in kilograms) of a dinosaur that walked on two feet can be modeled by

$$m = (0.00016)C^{2.73}$$

where *C* is the circumference (in millimeters) of the dinosaur's femur. The mass of a *Tyrannosaurus rex* was 4000 kilograms. Use a calculator to approximate the circumference of its femur.

6.2 Lesson

Core Vocabulary

*n*th root of *a*, *p. 300*
radical, *p. 300*
index of a radical, *p. 300*

Previous
square root

What You Will Learn

- Find *n*th roots.
- Evaluate expressions with rational exponents.
- Solve real-life problems involving rational exponents.

Finding *n*th Roots

You can extend the concept of a square root to other types of roots. For example, 2 is a cube root of 8 because $2^3 = 8$, and 3 is a fourth root of 81 because $3^4 = 81$. In general, for an integer n greater than 1, if $b^n = a$, then b is an **nth root of a**. An nth root of a is written as $\sqrt[n]{a}$, where the expression $\sqrt[n]{a}$ is called a **radical** and n is the **index** of the radical.

You can also write an nth root of a as a power of a. If you assume the Power of a Power Property applies to rational exponents, then the following is true.

$$(a^{1/2})^2 = a^{(1/2) \cdot 2} = a^1 = a$$
$$(a^{1/3})^3 = a^{(1/3) \cdot 3} = a^1 = a$$
$$(a^{1/4})^4 = a^{(1/4) \cdot 4} = a^1 = a$$

Because $a^{1/2}$ is a number whose square is a, you can write $\sqrt{a} = a^{1/2}$. Similarly, $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer n greater than 1.

READING

$\pm\sqrt[n]{a}$ represents both the positive and negative *n*th roots of *a*.

Core Concept

Real *n*th Roots of *a*

Let n be an integer greater than 1, and let a be a real number.

- If n is odd, then a has one real nth root: $\sqrt[n]{a} = a^{1/n}$
- If n is even and $a > 0$, then a has two real nth roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$
- If n is even and $a = 0$, then a has one real nth root: $\sqrt[n]{0} = 0$
- If n is even and $a < 0$, then a has no real nth roots.

The nth roots of a number may be real numbers or *imaginary numbers*. You will study imaginary numbers in a future course.

EXAMPLE 1 Finding *n*th Roots

Find the indicated real nth root(s) of a.

a. $n = 3, a = -27$

b. $n = 4, a = 16$

SOLUTION

a. The index $n = 3$ is odd, so -27 has one real cube root. Because $(-3)^3 = -27$, the cube root of -27 is $\sqrt[3]{-27} = -3$, or $(-27)^{1/3} = -3$.

b. The index $n = 4$ is even, and $a > 0$. So, 16 has two real fourth roots. Because $2^4 = 16$ and $(-2)^4 = 16$, the fourth roots of 16 are $\pm\sqrt[4]{16} = \pm 2$, or $\pm 16^{1/4} = \pm 2$.

Monitoring Progress

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Find the indicated real *n*th root(s) of *a*.

1. $n = 3, a = -125$

2. $n = 6, a = 64$

Evaluating Expressions with Rational Exponents

Recall that the radical $\sqrt{a}$ indicates the positive square root of a. Similarly, an nth root of a, $\sqrt[n]{a}$, with an *even* index indicates the positive nth root of a.

REMEMBER

The expression under the radical sign is the radicand.

EXAMPLE 2 Evaluating *n*th Root Expressions

Evaluate each expression.

a. $\sqrt[3]{-8}$ **b.** $-\sqrt[3]{8}$ **c.** $16^{1/4}$ **d.** $(-16)^{1/4}$

SOLUTION

a. $\sqrt[3]{-8} = \sqrt[3]{(-2) \cdot (-2) \cdot (-2)}$	Rewrite the expression showing factors.
$= -2$	Evaluate the cube root.
b. $-\sqrt[3]{8} = -\left(\sqrt[3]{2 \cdot 2 \cdot 2}\right)$	Rewrite the expression showing factors.
$= -(2)$	Evaluate the cube root.
$= -2$	Simplify.
c. $16^{1/4} = \sqrt[4]{16}$	Rewrite the expression in radical form.
$= \sqrt[4]{2 \cdot 2 \cdot 2 \cdot 2}$	Rewrite the expression showing factors.
$= 2$	Evaluate the fourth root.

d. $(-16)^{1/4}$ is not a real number because there is no real number that can be multiplied by itself four times to produce -16.

A rational exponent does not have to be of the form $1/n$. Other rational numbers such as $3/2$ can also be used as exponents. You can use the properties of exponents to evaluate or simplify expressions involving rational exponents.

STUDY TIP

You can rewrite $27^{2/3}$ as $27^{(1/3) \cdot 2}$ and then use the Power of a Power Property to show that

$27^{(1/3) \cdot 2} = (27^{1/3})^2$.

Core Concept

Rational Exponents

Let $a^{1/n}$ be an nth root of a, and let m be a positive integer.

Algebra $a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m$

Numbers $27^{2/3} = (27^{1/3})^2 = \left(\sqrt[3]{27}\right)^2$

EXAMPLE 3 Evaluating Expressions with Rational Exponents

Evaluate (a) $16^{3/4}$ and (b) $27^{4/3}$.

SOLUTION

a. $16^{3/4} = (16^{1/4})^3$	Rational exponents	**b.** $27^{4/3} = (27^{1/3})^4$
$= 2^3$	Evaluate the nth root.	$= 3^4$
$= 8$	Evaluate the power.	$= 81$

Monitoring Progress

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Evaluate the expression.

3. $\sqrt[3]{-125}$ **4.** $(-64)^{2/3}$ **5.** $9^{5/2}$ **6.** $256^{3/4}$

Solving Real-Life Problems

EXAMPLE 4 Solving a Real-Life Problem

Volume = 113 cubic feet

The radius r of a sphere is given by the equation $r = \left(\frac{3V}{4\pi}\right)^{1/3}$, where V is the volume of the sphere. Find the radius of the beach ball to the nearest foot. Use 3.14 for π.

SOLUTION

1. **Understand the Problem** You know the equation that represents the radius of a sphere in terms of its volume. You are asked to find the radius for a given volume.
2. **Make a Plan** Substitute the given volume into the equation. Then evaluate to find the radius.
3. **Solve the Problem**

$$r = \left(\frac{3V}{4\pi}\right)^{1/3} \qquad \text{Write the equation.}$$

$$= \left(\frac{3(113)}{4(3.14)}\right)^{1/3} \qquad \text{Substitute 113 for } V \text{ and 3.14 for } \pi.$$

$$= \left(\frac{339}{12.56}\right)^{1/3} \qquad \text{Multiply.}$$

$$\approx 3 \qquad \text{Use a calculator.}$$

▶ The radius of the beach ball is about 3 feet.

4. **Look Back** To check that your answer is reasonable, compare the size of the ball to the size of the woman pushing the ball. The ball appears to be slightly taller than the woman. The average height of a woman is between 5 and 6 feet. So, a radius of 3 feet, or height of 6 feet, seems reasonable for the beach ball.

EXAMPLE 5 Solving a Real-Life Problem

To calculate the annual inflation rate r (in decimal form) of an item that increases in value from P to F over a period of n years, you can use the equation $r = \left(\frac{F}{P}\right)^{1/n} - 1$. Find the annual inflation rate to the nearest tenth of a percent of a house that increases in value from \$200,000 to \$235,000 over a period of 5 years.

SOLUTION

$$r = \left(\frac{F}{P}\right)^{1/n} - 1 \qquad \text{Write the equation.}$$

$$= \left(\frac{235{,}000}{200{,}000}\right)^{1/5} - 1 \qquad \text{Substitute 235,000 for } F\text{, 200,000 for } P\text{, and 5 for } n.$$

$$= 1.175^{1/5} - 1 \qquad \text{Divide.}$$

$$\approx 0.03278 \qquad \text{Use a calculator.}$$

▶ The annual inflation rate is about 3.3%.

REMEMBER

To write a decimal as a percent, move the decimal point two places to the right. Then add a percent symbol.

Monitoring Progress

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7. **WHAT IF?** In Example 4, the volume of the beach ball is 17,000 cubic inches. Find the radius to the nearest inch. Use 3.14 for π.
8. The average cost of college tuition increases from \$8500 to \$13,500 over a period of 8 years. Find the annual inflation rate to the nearest tenth of a percent.

6.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. WRITING Explain how to evaluate $81^{1/4}$.

2. WHICH ONE DOESN'T BELONG? Which expression does *not* belong with the other three? Explain your reasoning.

$(\sqrt[3]{27})^2$ $27^{2/3}$ 3^2 $(\sqrt[2]{27})^3$

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, rewrite the expression in rational exponent form.

3. $\sqrt{10}$

4. $\sqrt[5]{34}$

In Exercises 5 and 6, rewrite the expression in radical form.

5. $15^{1/3}$

6. $140^{1/8}$

In Exercises 7–10, find the indicated real *n*th root(s) of *a*. *(See Example 1.)*

7. $n = 2, a = 36$

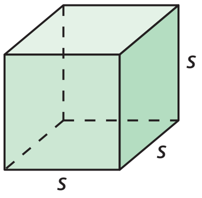
8. $n = 4, a = 81$

9. $n = 3, a = 1000$

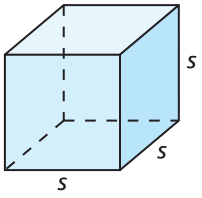
10. $n = 9, a = -512$

MATHEMATICAL CONNECTIONS **In Exercises 11 and 12, find the dimensions of the cube. Check your answer.**

11. Volume = 64 in.^{3}



12. Volume = 216 cm^3



In Exercises 13–18, evaluate the expression. *(See Example 2.)*

13. $\sqrt[4]{256}$

14. $\sqrt[3]{-216}$

15. $\sqrt[3]{-343}$

16. $-\sqrt[5]{1024}$

17. $128^{1/7}$

18. $(-64)^{1/2}$

In Exercises 19 and 20, rewrite the expression in rational exponent form.

19. $(\sqrt[5]{8})^4$

20. $(\sqrt[5]{-21})^6$

In Exercises 21 and 22, rewrite the expression in radical form.

21. $(-4)^{2/7}$

22. $9^{5/2}$

In Exercises 23–28, evaluate the expression. *(See Example 3.)*

23. $32^{3/5}$

24. $125^{2/3}$

25. $(-36)^{3/2}$

26. $(-243)^{2/5}$

27. $(-128)^{5/7}$

28. $343^{4/3}$

29. ERROR ANALYSIS Describe and correct the error in rewriting the expression in rational exponent form.

✗ $(\sqrt[3]{2})^4 = 2^{3/4}$

30. ERROR ANALYSIS Describe and correct the error in evaluating the expression.

✗
$$\begin{aligned}(-81)^{3/4} &= [(-81)^{1/4}]^3\\ &= (-3)^3\\ &= -27\end{aligned}$$

In Exercises 31–34, evaluate the expression.

31. $\left(\frac{1}{1000}\right)^{1/3}$

32. $\left(\frac{1}{64}\right)^{1/6}$

33. $(27)^{-2/3}$

34. $(9)^{-5/2}$

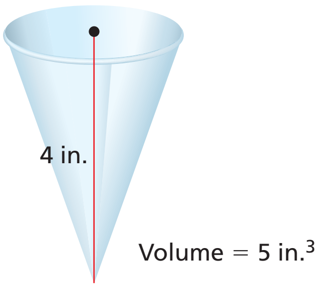
35. **PROBLEM SOLVING** A math club is having a bake sale. Find the area of the bake sale sign.

36. **PROBLEM SOLVING** The volume of a cube-shaped box is 27^5 cubic millimeters. Find the length of one side of the box.

37. **MODELING WITH MATHEMATICS** The radius r of the base of a cone is given by the equation

$$r = \left(\frac{3V}{\pi h}\right)^{1/2}$$

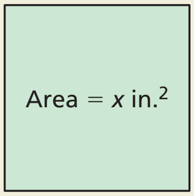
where V is the volume of the cone and h is the height of the cone. Find the radius of the paper cup to the nearest inch. Use 3.14 for π. *(See Example 4.)*



38. **MODELING WITH MATHEMATICS** The volume of a sphere is given by the equation $V = \frac{1}{6\sqrt{\pi}}S^{3/2}$, where S is the surface area of the sphere. Find the volume of a sphere, to the nearest cubic meter, that has a surface area of 60 square meters. Use 3.14 for π.

39. **WRITING** Explain how to write $(\sqrt[n]{a}\,)^m$ in rational exponent form.

40. **HOW DO YOU SEE IT?** Write an expression in rational exponent form that represents the side length of the square.



In Exercises 41 and 42, use the formula $r = \left(\frac{F}{P}\right)^{1/n} - 1$ to find the annual inflation rate to the nearest tenth of a percent. *(See Example 5.)*

41. A farm increases in value from \$800,000 to \$1,100,000 over a period of 6 years.

42. The cost of a gallon of gas increases from \$1.46 to \$3.53 over a period of 10 years.

43. **REASONING** For what values of x is $x = x^{1/5}$?

44. **MAKING AN ARGUMENT** Your friend says that for a real number a and a positive integer n, the value of $\sqrt[n]{a}$ is always positive and the value of $-\sqrt[n]{a}$ is always negative. Is your friend correct? Explain.

In Exercises 45–48, simplify the expression.

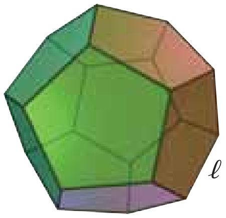
45. $(y^{1/6})^3 \cdot \sqrt{x}$

46. $(y \cdot y^{1/3})^{3/2}$

47. $x \cdot \sqrt[3]{y^6} + y^2 \cdot \sqrt[3]{x^3}$

48. $(x^{1/3} \cdot y^{1/2})^9 \cdot \sqrt{y}$

49. **PROBLEM SOLVING** The formula for the volume of a regular dodecahedron is $V \approx 7.66\ell^3$, where ℓ is the length of an edge. The volume of the dodecahedron is 20 cubic feet. Estimate the edge length.



50. **THOUGHT PROVOKING** Find a formula (for instance, from geometry or physics) that contains a radical. Rewrite the formula using rational exponents.

ABSTRACT REASONING **In Exercises 51–56, let x be a nonnegative real number. Determine whether the statement is *always*, *sometimes*, or *never* true. Justify your answer.**

51. $(x^{1/3})^3 = x$

52. $x^{1/3} = x^{-3}$

53. $x^{1/3} = \sqrt[3]{x}$

54. $x^{1/3} = x^3$

55. $\frac{x^{2/3}}{x^{1/3}} = \sqrt[3]{x}$

56. $x = x^{1/3} \cdot x^3$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the function when $x = -3$, 0, and 8. *(Section 3.3)*

57. $f(x) = 2x - 10$

58. $w(x) = -5x - 1$

59. $h(x) = 13 - x$

60. $g(x) = 8x + 16$

6.3 Exponential Functions

Essential Question What are some of the characteristics of the graph of an exponential function?

EXPLORATION 1 Exploring an Exponential Function

Work with a partner. Copy and complete each table for the *exponential function* $y = 16(2)^x$. In each table, what do you notice about the values of x? What do you notice about the values of y?

x	$y = 16(2)^x$
0	
1	
2	
3	
4	
5	

x	$y = 16(2)^x$
0	
2	
4	
6	
8	
10	

JUSTIFYING CONCLUSIONS

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2 Exploring an Exponential Function

Work with a partner. Repeat Exploration 1 for the exponential function $y = 16\left(\frac{1}{2}\right)^x$. Do you think the statement below is true for *any* exponential function? Justify your answer.

> *"As the independent variable x changes by a constant amount, the dependent variable y is multiplied by a constant factor."*

EXPLORATION 3 Graphing Exponential Functions

Work with a partner. Sketch the graphs of the functions given in Explorations 1 and 2. How are the graphs similar? How are they different?

Communicate Your Answer

4. What are some of the characteristics of the graph of an exponential function?

5. Sketch the graph of each exponential function. Does each graph have the characteristics you described in Question 4? Explain your reasoning.

a. $y = 2^x$ **b.** $y = 2(3)^x$ **c.** $y = 3(1.5)^x$

d. $y = \left(\frac{1}{2}\right)^x$ **e.** $y = 3\left(\frac{1}{2}\right)^x$ **f.** $y = 2\left(\frac{3}{4}\right)^x$

6.3 Lesson

Core Vocabulary

exponential function, *p. 306*

Previous

independent variable

dependent variable

parent function

What You Will Learn

- Identify and evaluate exponential functions.
- Graph exponential functions.
- Solve real-life problems involving exponential functions.

Identifying and Evaluating Exponential Functions

An **exponential function** is a nonlinear function of the form $y = ab^x$, where $a \neq 0$, $b \neq 1$, and $b > 0$. As the independent variable x changes by a constant amount, the dependent variable y is multiplied by a constant factor, which means consecutive y-values form a constant ratio.

EXAMPLE 1 Identifying Functions

Does each table represent a *linear* or an *exponential* function? Explain.

a.

x	0	1	2	3
y	2	4	6	8

b.

x	0	1	2	3
y	4	8	16	32

SOLUTION

a.

+ 1 + 1 + 1

x	0	1	2	3
y	2	4	6	8

+ 2 + 2 + 2

- As x increases by 1, y increases by 2. The rate of change is constant. So, the function is linear.

b.

+ 1 + 1 + 1

x	0	1	2	3
y	4	8	16	32

× 2 × 2 × 2

- As x increases by 1, y is multiplied by 2. So, the function is exponential.

STUDY TIP

In Example 1b, consecutive y-values form a constant ratio.

$$\frac{8}{4} = 2, \frac{16}{8} = 2, \frac{32}{16} = 2$$

EXAMPLE 2 Evaluating Exponential Functions

Evaluate each function for the given value of x.

a. $y = -2(5)^x;\ x = 3$

b. $y = 3(0.5)^x;\ x = -2$

SOLUTION

a. $y = -2(5)^x$ Write the function.

$= -2(5)^3$ Substitute for x.

$= -2(125)$ Evaluate the power.

$= -250$ Multiply.

b. $y = 3(0.5)^x$

$= 3(0.5)^{-2}$

$= 3(4)$

$= 12$

Monitoring Progress

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Does the table represent a *linear* or an *exponential* function? Explain.

1.

x	0	1	2	3
y	8	4	2	1

2.

x	−4	0	4	8
y	1	0	−1	−2

Evaluate the function when $x = -2$, 0, and $\frac{1}{2}$.

3. $y = 2(9)^x$

4. $y = 1.5(2)^x$

Graphing Exponential Functions

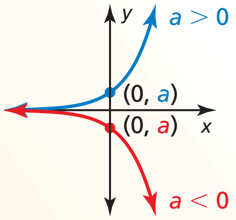
The graph of a function $y = ab^x$ is a vertical stretch or shrink by a factor of $|a|$ of the graph of the parent function $y = b^x$. When $a < 0$, the graph is also reflected in the x-axis. The y-intercept of the graph of $y = ab^x$ is a.

STUDY TIP

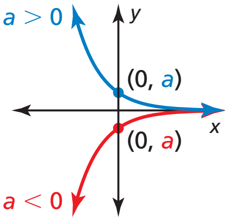
The graph of $y = ab^x$ approaches the x-axis but never intersects it.

Core Concept

Graphing $y = ab^x$ When $b > 1$



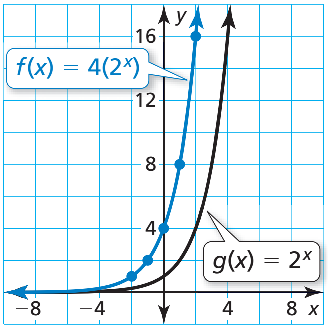
Graphing $y = ab^x$ When $0 < b < 1$



EXAMPLE 3 Graphing $y = ab^x$ When $b > 1$

Graph $f(x) = 4(2)^x$. Compare the graph to the graph of the parent function. Describe the domain and range of f.

SOLUTION



Step 1 Make a table of values.

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

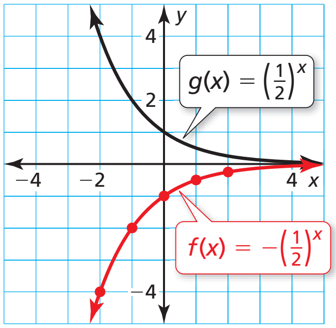
x	−2	−1	0	1	2
$f(x)$	1	2	4	8	16

▶ The parent function is $g(x) = 2^x$. The graph of f is a vertical stretch by a factor of 4 of the graph of g. The y-intercept of the graph of f, 4, is above the y-intercept of the graph of g, 1. From the graph of f, you can see that the domain is all real numbers and the range is $y > 0$.

EXAMPLE 4 Graphing $y = ab^x$ When $0 < b < 1$

Graph $f(x) = -\left(\frac{1}{2}\right)^x$. Compare the graph to the graph of the parent function. Describe the domain and range of f.

SOLUTION



Step 1 Make a table of values.

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

x	−2	−1	0	1	2
$f(x)$	−4	−2	−1	$-\frac{1}{2}$	$-\frac{1}{4}$

▶ The parent function is $g(x) = \left(\frac{1}{2}\right)^x$. The graph of f is a reflection in the x-axis of the graph of g. The y-intercept of the graph of f, −1, is below the y-intercept of the graph of g, 1. From the graph of f, you can see that the domain is all real numbers and the range is $y < 0$.

Monitoring Progress

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Graph the function. Compare the graph to the graph of the parent function. Describe the domain and range of f.

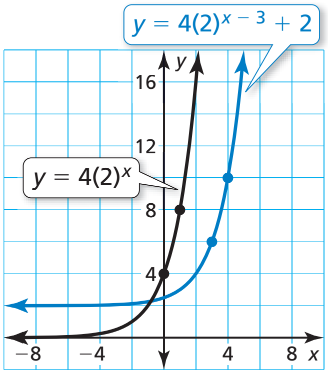
5. $f(x) = -2(4)^x$

6. $f(x) = 2\left(\frac{1}{4}\right)^x$

To graph a function of the form $y = ab^{x-h} + k$, begin by graphing $y = ab^x$. Then translate the graph horizontally h units and vertically k units.

EXAMPLE 5 Graphing $y = ab^{x-h} + k$

Graph $y = 4(2)^{x-3} + 2$. Describe the domain and range.



SOLUTION

Step 1 Graph $y = 4(2)^x$. This is the same function that is in Example 3, which passes through (0, 4) and (1, 8).

Step 2 Translate the graph 3 units right and 2 units up. The graph passes through (3, 6) and (4, 10).

Notice that the graph approaches the line $y = 2$ but does not intersect it.

▶ From the graph, you can see that the domain is all real numbers and the range is $y > 2$.

EXAMPLE 6 Comparing Exponential Functions

An exponential function g models a relationship in which the dependent variable is multiplied by 1.5 for every 1 unit the independent variable x increases. Graph g when $g(0) = 4$. Compare g and the function f from Example 3 over the interval $x = 0$ to $x = 2$.

SOLUTION

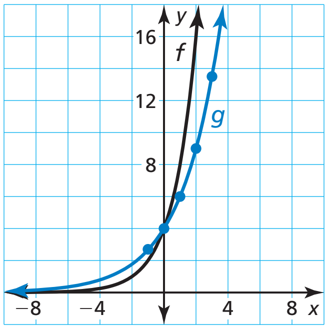
You know (0, 4) is on the graph of g. To find points to the right of (0, 4), multiply $g(x)$ by 1.5 for every 1 unit increase in x. To find points to the left of (0, 4), divide $g(x)$ by 1.5 for every 1 unit decrease in x.

Step 1 Make a table of values.

x	−1	0	1	2	3
***g*(*x*)**	2.7	4	6	9	13.5

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



STUDY TIP

Note that f is increasing faster than g to the right of $x = 0$.

▶ Both functions have the same value when $x = 0$, but the value of f is greater than the value of g over the rest of the interval.

Monitoring Progress

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Graph the function. Describe the domain and range.

7. $y = -2(3)^{x+2} - 1$

8. $f(x) = (0.25)^x + 3$

9. **WHAT IF?** In Example 6, the dependent variable of g is multiplied by 3 for every 1 unit the independent variable x increases. Graph g when $g(0) = 4$. Compare g and the function f from Example 3 over the interval $x = 0$ to $x = 2$.

Solving Real-Life Problems

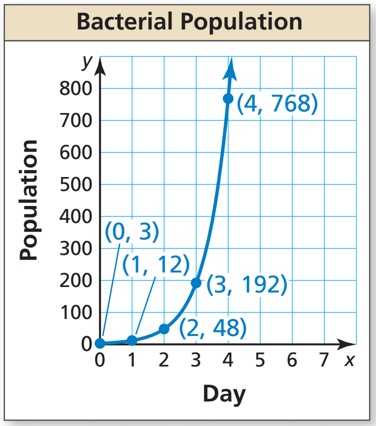
For an exponential function of the form $y = ab^x$, the y-values change by a factor of b as x increases by 1. You can use this fact to write an exponential function when you know the y-intercept, a. The table represents the exponential function $y = 2(5)^x$.

+1 +1 +1 +1

x	0	1	2	3	4
y	2	10	50	250	1250

×5 ×5 ×5 ×5

EXAMPLE 7 Modeling with Mathematics



The graph represents a bacterial population y after x days.

a. Write an exponential function that represents the population.

b. Find the population after 12 hours and after 5 days.

SOLUTION

1. **Understand the Problem** You have a graph of the population that shows some data points. You are asked to write an exponential function that represents the population and find the population after different amounts of time.

2. **Make a Plan** Use the graph to make a table of values. Use the table and the y-intercept to write an exponential function. Then evaluate the function to find the populations.

3. **Solve the Problem**

   a. Use the graph to make a table of values.

+1 +1 +1 +1

x	0	1	2	3	4
y	3	12	48	192	768

×4 ×4 ×4 ×4

The y-intercept is 3. The y-values increase by a factor of 4 as x increases by 1.

▶ So, the population can be modeled by $y = 3(4)^x$.

b. ***Population after 12 hours*** ***Population after 5 days***

12 hours = $\frac{1}{2}$ day

$y = 3(4)^x$	Write the function.	$y = 3(4)^x$
$= 3(4)^{1/2}$	Substitute for x.	$= 3(4)^5$
$= 3(2)$	Evaluate the power.	$= 3(1024)$
$= 6$	Multiply.	$= 3072$

▶ There are 6 bacteria after 12 hours and 3072 bacteria after 5 days.

4. **Look Back** The graph resembles an exponential function of the form $y = ab^x$, where $b > 1$ and $a > 0$. So, the exponential function $y = 3(4)^x$ is reasonable.

Monitoring Progress

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10. A bacterial population y after x days can be represented by an exponential function whose graph passes through (0, 100) and (1, 200). (a) Write a function that represents the population. (b) Find the population after 6 days. (c) Does this bacterial population grow faster than the bacterial population in Example 7? Explain.

6.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **OPEN-ENDED** Sketch an increasing exponential function whose graph has a y-intercept of 2.

2. **REASONING** Why is a the y-intercept of the graph of the function $y = ab^x$?

3. **WRITING** Compare the graph of $y = 2(5)^x$ with the graph of $y = 5^x$.

4. **WHICH ONE DOESN'T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

$y = 3^x$ $\quad f(x) = 2(4)^x$ $\quad f(x) = (-3)^x$ $\quad y = 5(3)^x$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, determine whether the equation represents an exponential function. Explain.

5. $y = 4(7)^x$

6. $y = -6x$

7. $y = 2x^3$

8. $y = -3^x$

9. $y = 9(-5)^x$

10. $y = \frac{1}{2}(1)^x$

In Exercises 11–14, determine whether the table represents a *linear* or an *exponential* function. Explain. *(See Example 1.)*

11.

x	y
1	−2
2	0
3	2
4	4

12.

x	y
1	6
2	12
3	24
4	48

13.

x	−1	0	1	2	3
y	0.25	1	4	16	64

14.

x	−3	0	3	6	9
y	10	1	−8	−17	−26

In Exercises 15–20, evaluate the function for the given value of x. *(See Example 2.)*

15. $y = 3^x;\ x = 2$

16. $f(x) = 3(2)^x;\ x = -1$

17. $y = -4(5)^x;\ x = 2$

18. $f(x) = 0.5^x;\ x = -3$

19. $f(x) = \frac{1}{3}(6)^x;\ x = 3$

20. $y = \frac{1}{4}(4)^x;\ x = \frac{3}{2}$

USING STRUCTURE In Exercises 21–24, match the function with its graph.

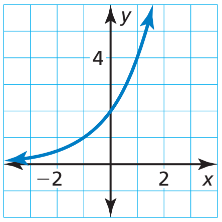
21. $f(x) = 2(0.5)^x$

22. $y = -2(0.5)^x$

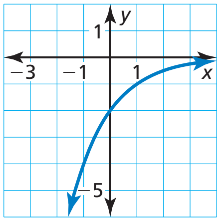
23. $y = 2(2)^x$

24. $f(x) = -2(2)^x$

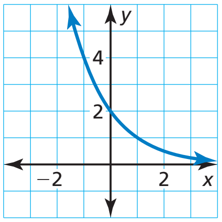
A.



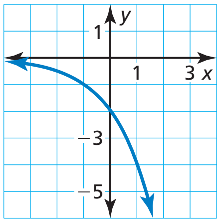
B.



C.



D.



In Exercises 25–30, graph the function. Compare the graph to the graph of the parent function. Describe the domain and range of f. *(See Examples 3 and 4.)*

25. $f(x) = 3(0.5)^x$

26. $f(x) = -4^x$

27. $f(x) = -2(7)^x$

28. $f(x) = 6\left(\frac{1}{3}\right)^x$

29. $f(x) = \frac{1}{2}(8)^x$

30. $f(x) = \frac{3}{2}(0.25)^x$

In Exercises 31–36, graph the function. Describe the domain and range. *(See Example 5.)*

31. $f(x) = 3^x - 1$

32. $f(x) = 4^{x+3}$

33. $y = 5^{x-2} + 7$

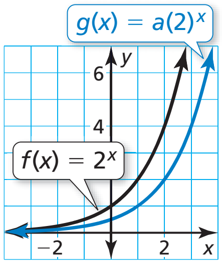
34. $y = -\left(\frac{1}{2}\right)^{x+1} - 3$

35. $y = -8(0.75)^{x+2} - 2$

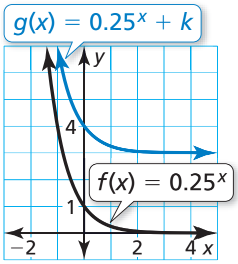
36. $f(x) = 3(6)^{x-1} - 5$

In Exercises 37–40, compare the graphs. Find the value of h, k, or a.

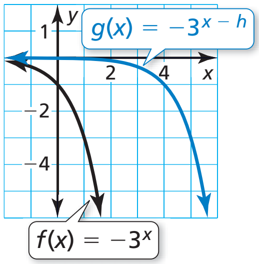
37.



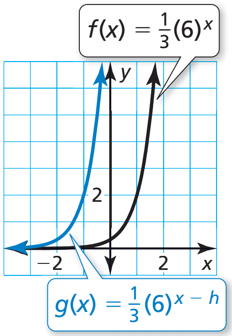
38.



39.



40.

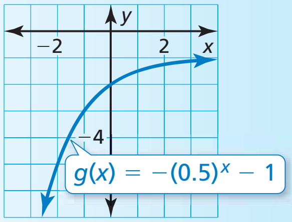


41. ERROR ANALYSIS Describe and correct the error in evaluating the function.

$$g(x) = 6(0.5)^x;\ x = -2$$
$$g(-2) = 6(0.5)^{-2}$$
$$= 3^{-2}$$
$$= \frac{1}{9}$$

42. ERROR ANALYSIS Describe and correct the error in finding the domain and range of the function.

The domain is all real numbers, and the range is $y < 0$.



In Exercises 43 and 44, graph the function with the given description. Compare the function to $f(x) = 0.5(4)^x$ over the interval $x = 0$ to $x = 2$. *(See Example 6.)*

43. An exponential function g models a relationship in which the dependent variable is multiplied by 2.5 for every 1 unit the independent variable x increases. The value of the function at 0 is 8.

44. An exponential function h models a relationship in which the dependent variable is multiplied by $\frac{1}{2}$ for every 1 unit the independent variable x increases. The value of the function at 0 is 32.

45. MODELING WITH MATHEMATICS You graph an exponential function on a calculator. You zoom in repeatedly to 25% of the screen size. The function $y = 0.25^x$ represents the percent (in decimal form) of the original screen display that you see, where x is the number of times you zoom in.

a. Graph the function. Describe the domain and range.

b. Find and interpret the y-intercept.

c. You zoom in twice. What percent of the original screen do you see?

46. MODELING WITH MATHEMATICS A population y of coyotes in a national park triples every 20 years. The function $y = 15(3)^x$ represents the population, where x is the number of 20-year periods.

a. Graph the function. Describe the domain and range.

b. Find and interpret the y-intercept.

c. How many coyotes are in the national park in 40 years?

In Exercises 47–50, write an exponential function represented by the table or graph. *(See Example 7.)*

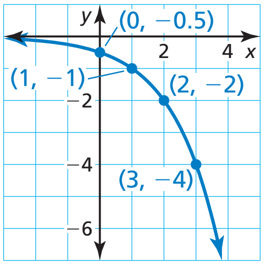
47.

x	0	1	2	3
y	2	14	98	686

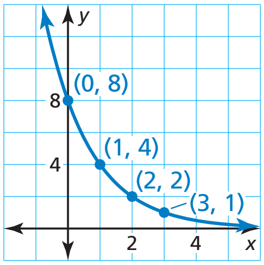
48.

x	0	1	2	3
y	−50	−10	−2	−0.4

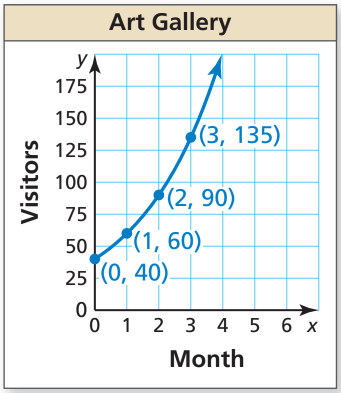
49.



50.



51. MODELING WITH MATHEMATICS The graph represents the number y of visitors to a new art gallery after x months.



a. Write an exponential function that represents this situation.

b. Approximate the number of visitors after 5 months.

52. PROBLEM SOLVING A sales report shows that 3300 gas grills were purchased from a chain of hardware stores last year. The store expects grill sales to increase 6% each year. About how many grills does the store expect to sell in Year 6? Use an equation to justify your answer.

53. WRITING Graph the function $f(x) = -2^x$. Then graph $g(x) = -2^x - 3$. How are the y-intercept, domain, and range affected by the translation?

54. MAKING AN ARGUMENT Your friend says that the table represents an exponential function because y is multiplied by a constant factor. Is your friend correct? Explain.

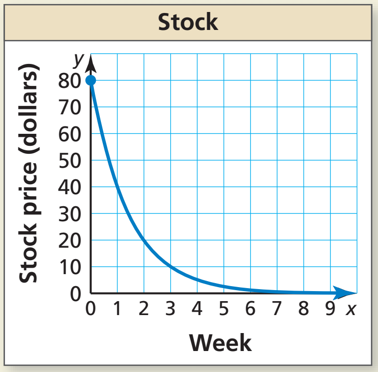
x	0	1	3	6
y	2	10	50	250

55. WRITING Describe the effect of a on the graph of $y = a \cdot 2^x$ when a is positive and when a is negative.

56. OPEN-ENDED Write a function whose graph is a horizontal translation of the graph of $h(x) = 4^x$.

57. USING STRUCTURE The graph of g is a translation 4 units up and 3 units right of the graph of $f(x) = 5^x$. Write an equation for g.

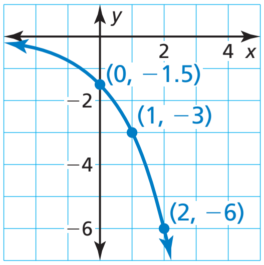
58. HOW DO YOU SEE IT? The exponential function $y = V(x)$ represents the projected value of a stock x weeks after a corporation loses an important legal battle. The graph of the function is shown.



a. After how many weeks will the stock be worth $20?

b. Describe the change in the stock price from Week 1 to Week 3.

59. USING GRAPHS The graph represents the exponential function f. Find $f(7)$.



60. THOUGHT PROVOKING Write a function of the form $y = ab^x$ that represents a real-life population. Explain the meaning of each of the constants a and b in the real-life context.

61. REASONING Let $f(x) = ab^x$. Show that when x is increased by a constant k, the quotient $\dfrac{f(x+k)}{f(x)}$ is always the same regardless of the value of x.

62. PROBLEM SOLVING A function g models a relationship in which the dependent variable is multiplied by 4 for every 2 units the independent variable increases. The value of the function at 0 is 5. Write an equation that represents the function.

63. PROBLEM SOLVING Write an exponential function f so that the slope from the point $(0, f(0))$ to the point $(2, f(2))$ is equal to 12.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the percent as a decimal. *(Skills Review Handbook)*

64. 4%

65. 35%

66. 128%

67. 250%

6.4 Exponential Growth and Decay

Essential Question What are some of the characteristics of exponential growth and exponential decay functions?

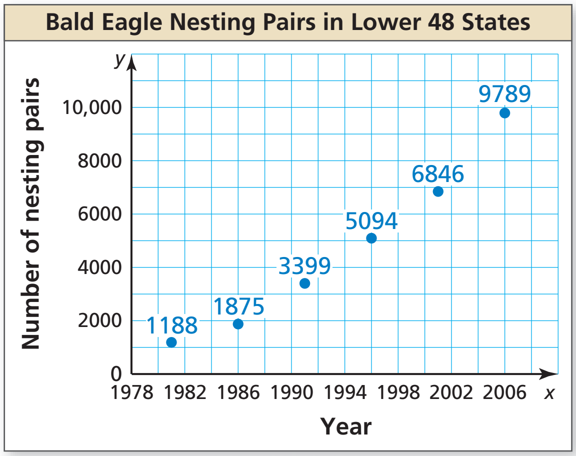
EXPLORATION 1 Predicting a Future Event

Work with a partner. It is estimated, that in 1782, there were about 100,000 nesting pairs of bald eagles in the United States. By the 1960s, this number had dropped to about 500 nesting pairs. In 1967, the bald eagle was declared an endangered species in the United States. With protection, the nesting pair population began to increase. Finally, in 2007, the bald eagle was removed from the list of endangered and threatened species.

Describe the pattern shown in the graph. Is it exponential growth? Assume the pattern continues. When will the population return to that of the late 1700s? Explain your reasoning.

MODELING WITH MATHEMATICS

To be proficient in math, you need to apply the mathematics you know to solve problems arising in everyday life.



EXPLORATION 2 Describing a Decay Pattern

Work with a partner. A forensic pathologist was called to estimate the time of death of a person. At midnight, the body temperature was 80.5°F and the room temperature was a constant 60°F. One hour later, the body temperature was 78.5°F.

a. By what percent did the difference between the body temperature and the room temperature drop during the hour?

b. Assume that the original body temperature was 98.6°F. Use the percent decrease found in part (a) to make a table showing the decreases in body temperature. Use the table to estimate the time of death.

Communicate Your Answer

3. What are some of the characteristics of exponential growth and exponential decay functions?

4. Use the Internet or some other reference to find an example of each type of function. Your examples should be different than those given in Explorations 1 and 2.

a. exponential growth **b.** exponential decay

6.4 Lesson

Core Vocabulary

exponential growth, *p. 314*
exponential growth function, *p. 314*
exponential decay, *p. 315*
exponential decay function, *p. 315*
compound interest, *p. 317*

What You Will Learn

- Use and identify exponential growth and decay functions.
- Interpret and rewrite exponential growth and decay functions.
- Solve real-life problems involving exponential growth and decay.

Exponential Growth and Decay Functions

Exponential growth occurs when a quantity increases by the same factor over equal intervals of time.

Core Concept

Exponential Growth Functions

A function of the form $y = a(1 + r)^t$, where $a > 0$ and $r > 0$, is an **exponential growth function**.

$$\underset{\text{final amount}}{y} = \overset{\text{initial amount}}{a}\underbrace{(1 + \overset{\text{rate of growth (in decimal form)}}{r})}_{\text{growth factor}}{}^{\overset{\text{time}}{t}}$$

STUDY TIP

Notice that an exponential growth function is of the form $y = ab^x$, where b is replaced by $1 + r$ and x is replaced by t.

EXAMPLE 1 Using an Exponential Growth Function

The inaugural attendance of an annual music festival is 150,000. The attendance y increases by 8% each year.

a. Write an exponential growth function that represents the attendance after t years.

b. How many people will attend the festival in the fifth year? Round your answer to the nearest thousand.

SOLUTION

a. The initial amount is 150,000, and the rate of growth is 8%, or 0.08.

$y = a(1 + r)^t$ Write the exponential growth function.

$= 150{,}000(1 + 0.08)^t$ Substitute 150,000 for a and 0.08 for r.

$= 150{,}000(1.08)^t$ Add.

▶ The festival attendance can be represented by $y = 150{,}000(1.08)^t$.

b. The value $t = 4$ represents the fifth year because $t = 0$ represents the first year.

$y = 150{,}000(1.08)^t$ Write the exponential growth function.

$= 150{,}000(1.08)^4$ Substitute 4 for t.

$\approx 204{,}073$ Use a calculator.

▶ About 204,000 people will attend the festival in the fifth year.

Monitoring Progress

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1. A website has 500,000 members in 2010. The number y of members increases by 15% each year. (a) Write an exponential growth function that represents the website membership t years after 2010. (b) How many members will there be in 2016? Round your answer to the nearest ten thousand.

Exponential decay occurs when a quantity decreases by the same factor over equal intervals of time.

STUDY TIP

Notice that an exponential decay function is of the form $y = ab^x$, where b is replaced by $1 - r$ and x is replaced by t.

Core Concept

Exponential Decay Functions

A function of the form $y = a(1 - r)^t$, where $a > 0$ and $0 < r < 1$, is an **exponential decay function**.

$$\underset{\text{final amount}}{y} = \overset{\text{initial amount}}{a}\underbrace{(1 - \overset{\text{rate of decay (in decimal form)}}{r})}_{\text{decay factor}}{}^{\overset{\text{time}}{t}}$$

For exponential growth, the value inside the parentheses is greater than 1 because r is added to 1. For exponential decay, the value inside the parentheses is less than 1 because r is subtracted from 1.

EXAMPLE 2 Identifying Exponential Growth and Decay

Determine whether each table represents an *exponential growth function*, an *exponential decay function*, or *neither*.

a.

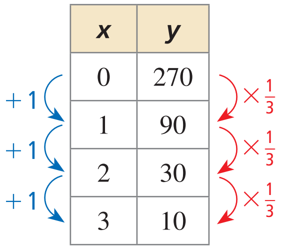
x	*y*
0	270
1	90
2	30
3	10

b.

x	0	1	2	3
y	5	10	20	40

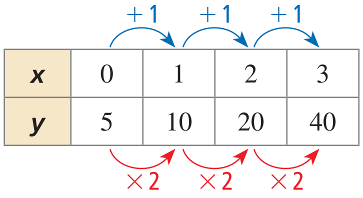
SOLUTION

a.



x	*y*
0	270
1	90
2	30
3	10

b.



x	0	1	2	3
y	5	10	20	40

▶ As x increases by 1, y is multiplied by $\frac{1}{3}$. So, the table represents an exponential decay function.

▶ As x increases by 1, y is multiplied by 2. So, the table represents an exponential growth function.

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Determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain.

2.

x	0	1	2	3
y	64	16	4	1

3.

x	1	3	5	7
y	4	11	18	25

Interpreting and Rewriting Exponential Functions

EXAMPLE 3 Interpreting Exponential Functions

Determine whether each function represents *exponential growth* or *exponential decay*. Identify the percent rate of change.

a. $y = 5(1.07)^t$ **b.** $f(t) = 0.2(0.98)^t$

SOLUTION

a. The function is of the form $y = a(1 + r)^t$, where $1 + r > 1$, so it represents exponential growth. Use the growth factor $1 + r$ to find the rate of growth.

$1 + r = 1.07$ Write an equation.

$r = 0.07$ Solve for *r*.

▶ So, the function represents exponential growth and the rate of growth is 7%.

b. The function is of the form $y = a(1 - r)^t$, where $1 - r < 1$, so it represents exponential decay. Use the decay factor $1 - r$ to find the rate of decay.

$1 - r = 0.98$ Write an equation.

$r = 0.02$ Solve for *r*.

▶ So, the function represents exponential decay and the rate of decay is 2%.

STUDY TIP

You can rewrite exponential expressions and functions using the properties of exponents. Changing the form of an exponential function can reveal important attributes of the function.

EXAMPLE 4 Rewriting Exponential Functions

Rewrite each function to determine whether it represents *exponential growth* or *exponential decay*.

a. $y = 100(0.96)^{t/4}$ **b.** $f(t) = (1.1)^{t-3}$

SOLUTION

a. $y = 100(0.96)^{t/4}$ Write the function.

$= 100(0.96^{1/4})^t$ Power of a Power Property

$\approx 100(0.99)^t$ Evaluate the power.

▶ So, the function represents exponential decay.

b. $f(t) = (1.1)^{t-3}$ Write the function.

$= \frac{(1.1)^t}{(1.1)^3}$ Quotient of Powers Property

$\approx 0.75(1.1)^t$ Evaluate the power and simplify.

▶ So, the function represents exponential growth.

Monitoring Progress

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Determine whether the function represents *exponential growth* or *exponential decay*. Identify the percent rate of change.

4. $y = 2(0.92)^t$ **5.** $f(t) = (1.2)^t$

Rewrite the function to determine whether it represents *exponential growth* or *exponential decay*.

6. $f(t) = 3(1.02)^{10t}$ **7.** $y = (0.95)^{t+2}$

Solving Real-Life Problems

Exponential growth functions are used in real-life situations involving *compound interest*. Although interest earned is expressed as an *annual* rate, the interest is usually compounded more frequently than once per year. So, the formula $y = a(1 + r)^t$ must be modified for compound interest problems.

STUDY TIP

For interest compounded yearly, you can substitute 1 for n in the formula to get $y = P(1 + r)^t$.

Core Concept

Compound Interest

Compound interest is the interest earned on the principal *and* on previously earned interest. The balance y of an account earning compound interest is

$$y = P\left(1 + \frac{r}{n}\right)^{nt}.$$

P = principal (initial amount)
r = annual interest rate (in decimal form)
t = time (in years)
n = number of times interest is compounded per year

EXAMPLE 5 Writing a Function

You deposit \$100 in a savings account that earns 6% annual interest compounded monthly. Write a function that represents the balance after t years.

SOLUTION

$$y = P\left(1 + \frac{r}{n}\right)^{nt}$$ Write the compound interest formula.

$$= 100\left(1 + \frac{0.06}{12}\right)^{12t}$$ Substitute 100 for P, 0.06 for r, and 12 for n.

$$= 100(1.005)^{12t}$$ Simplify.

EXAMPLE 6 Solving a Real-Life Problem

The table shows the balance of a money market account over time.

a. Write a function that represents the balance after t years.

b. Graph the functions from part (a) and from Example 5 in the same coordinate plane. Compare the account balances.

Year, t	Balance
0	\$100
1	\$110
2	\$121
3	\$133.10
4	\$146.41
5	\$161.05

SOLUTION

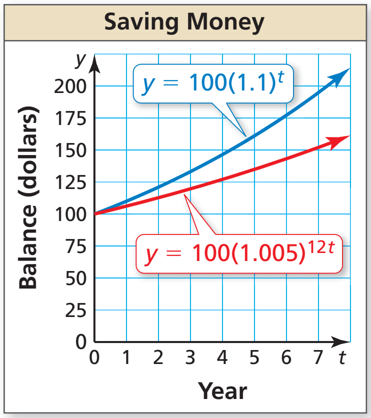
a. From the table, you know the initial balance is \$100, and it increases 10% each year. So, $P = 100$ and $r = 0.1$.

$$y = P(1 + r)^t$$ Write the compound interest formula when $n = 1$.

$$= 100(1 + 0.1)^t$$ Substitute 100 for P and 0.1 for r.

$$= 100(1.1)^t$$ Add.

b. The money market account earns 10% interest each year, and the savings account earns 6% interest each year. So, the balance of the money market account increases faster.



Monitoring Progress

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8. You deposit \$500 in a savings account that earns 9% annual interest compounded monthly. Write and graph a function that represents the balance y (in dollars) after t years.

EXAMPLE 7 Solving a Real-Life Problem

The value of a car is \$21,500. It loses 12% of its value every year. (a) Write a function that represents the value y (in dollars) of the car after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Use the graph to estimate the value of the car after 6 years.

STUDY TIP

In real life, the percent decrease in value of an asset is called the *depreciation rate*.

SOLUTION

1. **Understand the Problem** You know the value of the car and its annual percent decrease in value. You are asked to write a function that represents the value of the car over time and approximate the monthly percent decrease in value. Then graph the function and use the graph to estimate the value of the car in the future.

2. **Make a Plan** Use the initial amount and the annual percent decrease in value to write an exponential decay function. Note that the annual percent decrease represents the rate of decay. Rewrite the function using the properties of exponents to approximate the monthly percent decrease (rate of decay). Then graph the original function and use the graph to estimate the y-value when the t-value is 6.

3. **Solve the Problem**

   a. The initial value is \$21,500, and the rate of decay is 12%, or 0.12.

   $y = a(1 - r)^t$ Write the exponential decay function.

   $= 21{,}500(1 - 0.12)^t$ Substitute 21,500 for a and 0.12 for r.

   $= 21{,}500(0.88)^t$ Subtract.

   ▶ The value of the car can be represented by $y = 21{,}500(0.88)^t$.

   b. Use the fact that $t = \frac{1}{12}(12t)$ and the properties of exponents to rewrite the function in a form that reveals the monthly rate of decay.

   $y = 21{,}500(0.88)^t$ Write the original function.

   $= 21{,}500(0.88)^{(1/12)(12t)}$ Rewrite the exponent.

   $= 21{,}500(0.88^{1/12})^{12t}$ Power of a Power Property

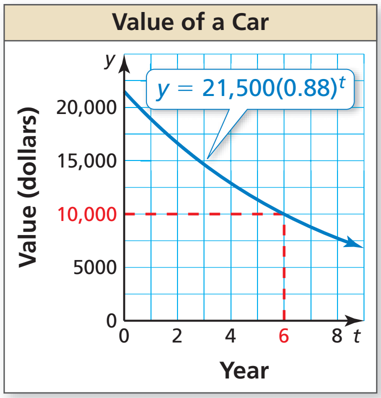
   $\approx 21{,}500(0.989)^{12t}$ Evaluate the power.

   Use the decay factor $1 - r \approx 0.989$ to find the rate of decay $r \approx 0.011$.

   ▶ So, the monthly percent decrease is about 1.1%.

   c. From the graph, you can see that the y-value is about 10,000 when $t = 6$.

   ▶ So, the value of the car is about \$10,000 after 6 years.



4. **Look Back** To check that the monthly percent decrease is reasonable, multiply it by 12 to see if it is close in value to the annual percent decrease of 12%.

   $1.1\% \times 12 = 13.2\%$ 13.2% is close to 12%, so 1.1% is reasonable.

   When you evaluate $y = 21{,}500(0.88)^t$ for $t = 6$, you get about \$9985. So, \$10,000 is a reasonable estimation.

Monitoring Progress

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9. **WHAT IF?** The car loses 9% of its value every year. (a) Write a function that represents the value y (in dollars) of the car after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Use the graph to estimate the value of the car after 12 years. Round your answer to the nearest thousand.

6.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** In the exponential growth function $y = a(1 + r)^t$, the quantity r is called the ________.

2. **VOCABULARY** What is the decay factor in the exponential decay function $y = a(1 - r)^t$?

3. **VOCABULARY** Compare exponential growth and exponential decay.

4. **WRITING** When does the function $y = ab^x$ represent exponential growth? exponential decay?

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, identify the initial amount a and the rate of growth r (as a percent) of the exponential function. Evaluate the function when $t = 5$. Round your answer to the nearest tenth.

5. $y = 350(1 + 0.75)^t$

6. $y = 10(1 + 0.4)^t$

7. $y = 25(1.2)^t$

8. $y = 12(1.05)^t$

9. $f(t) = 1500(1.074)^t$

10. $h(t) = 175(1.028)^t$

11. $g(t) = 6.72(2)^t$

12. $p(t) = 1.8^t$

In Exercises 13–16, write a function that represents the situation.

13. Sales of \$10,000 increase by 65% each year.

14. Your starting annual salary of \$35,000 increases by 4% each year.

15. A population of 210,000 increases by 12.5% each year.

16. An item costs \$4.50, and its price increases by 3.5% each year.

17. **MODELING WITH MATHEMATICS** The population of a city has been increasing by 2% annually. The sign shown is from the year 2000. *(See Example 1.)*

    a. Write an exponential growth function that represents the population t years after 2000.

    b. What will the population be in 2020? Round your answer to the nearest thousand.

18. **MODELING WITH MATHEMATICS** A young channel catfish weighs about 0.1 pound. During the next 8 weeks, its weight increases by about 23% each week.

    a. Write an exponential growth function that represents the weight of the catfish after t weeks during the 8-week period.

    b. About how much will the catfish weigh after 4 weeks? Round your answer to the nearest thousandth.

In Exercises 19–26, identify the initial amount a and the rate of decay r (as a percent) of the exponential function. Evaluate the function when $t = 3$. Round your answer to the nearest tenth.

19. $y = 575(1 - 0.6)^t$

20. $y = 8(1 - 0.15)^t$

21. $g(t) = 240(0.75)^t$

22. $f(t) = 475(0.5)^t$

23. $w(t) = 700(0.995)^t$

24. $h(t) = 1250(0.865)^t$

25. $y = \left(\frac{7}{8}\right)^t$

26. $y = 0.5\left(\frac{3}{4}\right)^t$

In Exercises 27–30, write a function that represents the situation.

27. A population of 100,000 decreases by 2% each year.

28. A \$900 sound system decreases in value by 9% each year.

29. A stock valued at \$100 decreases in value by 9.5% each year.

30. A company profit of \$20,000 decreases by 13.4% each year.

31. **ERROR ANALYSIS** The growth rate of a bacterial culture is 150% each hour. Initially, there are 10 bacteria. Describe and correct the error in finding the number of bacteria in the culture after 8 hours.

$b(t) = 10(1.5)^t$

$b(8) = 10(1.5)^8 \approx 256.3$

After 8 hours, there are about 256 bacteria in the culture.

32. **ERROR ANALYSIS** You purchase a car in 2010 for \$25,000. The value of the car decreases by 14% annually. Describe and correct the error in finding the value of the car in 2015.

$v(t) = 25{,}000(1.14)^t$

$v(5) = 25{,}000(1.14)^5 \approx 48{,}135$

The value of the car in 2015 is about \$48,000.

In Exercises 33–38, determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain. *(See Example 2.)*

33.

x	y
−1	50
0	10
1	2
2	0.4

34.

x	y
0	32
1	28
2	24
3	20

35.

x	y
0	35
1	29
2	23
3	17

36.

x	y
1	17
2	51
3	153
4	459

37.

x	y
5	2
10	8
15	32
20	128

38.

x	y
3	432
5	72
7	12
9	2

39. **ANALYZING RELATIONSHIPS** The table shows the value of a camper *t* years after it is purchased.

t	Value
1	\$37,000
2	\$29,600
3	\$23,680
4	\$18,944

a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.

b. What is the value of the camper after 5 years?

40. **ANALYZING RELATIONSHIPS** The table shows the total numbers of visitors to a website *t* days after it is online.

t	42	43	44	45
Visitors	11,000	12,100	13,310	14,641

a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.

b. How many people will have visited the website after it is online 47 days?

In Exercises 41–48, determine whether each function represents *exponential growth* or *exponential decay*. Identify the percent rate of change. *(See Example 3.)*

41. $y = 4(0.8)^t$

42. $y = 15(1.1)^t$

43. $y = 30(0.95)^t$

44. $y = 5(1.08)^t$

45. $r(t) = 0.4(1.06)^t$

46. $s(t) = 0.65(0.48)^t$

47. $g(t) = 2\left(\frac{5}{4}\right)^t$

48. $m(t) = \left(\frac{4}{5}\right)^t$

In Exercises 49–56, rewrite the function to determine whether it represents *exponential growth* or *exponential decay*. *(See Example 4.)*

49. $y = (0.9)^{t-4}$

50. $y = (1.4)^{t+8}$

51. $y = 2(1.06)^{9t}$

52. $y = 5(0.82)^{t/5}$

53. $x(t) = (1.45)^{t/2}$

54. $f(t) = 0.4(1.16)^{t-1}$

55. $b(t) = 4(0.55)^{t+3}$

56. $r(t) = (0.88)^{4t}$

In Exercises 57–60, write a function that represents the balance after *t* years. *(See Example 5.)*

57. \$2000 deposit that earns 5% annual interest compounded quarterly

58. \$1400 deposit that earns 10% annual interest compounded semiannually

59. \$6200 deposit that earns 8.4% annual interest compounded monthly

60. \$3500 deposit that earns 9.2% annual interest compounded quarterly

61. **PROBLEM SOLVING** The cross-sectional area of a tree 4.5 feet from the ground is called its *basal area*. The table shows the basal areas (in square inches) of Tree A over time. *(See Example 6.)*

Year, *t*	0	1	2	3	4
Basal area, *A*	120	132	145.2	159.7	175.7

Growth rate: 6%

Initial basal area: 154 in.^{2}

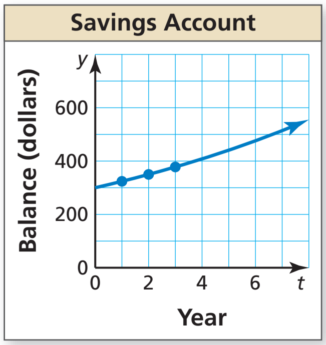
a. Write functions that represent the basal areas of the trees after *t* years.

b. Graph the functions from part (a) in the same coordinate plane. Compare the basal areas.

62. **PROBLEM SOLVING** You deposit \$300 into an investment account that earns 12% annual interest compounded quarterly. The graph shows the balance of a savings account over time.

a. Write functions that represent the balances of the accounts after *t* years.

b. Graph the functions from part (a) in the same coordinate plane. Compare the account balances.



63. **PROBLEM SOLVING** A city has a population of 25,000. The population is expected to increase by 5.5% annually for the next decade. *(See Example 7.)*

a. Write a function that represents the population *y* after *t* years.

b. Find the approximate monthly percent increase in population.

c. Graph the function from part (a). Use the graph to estimate the population after 4 years.

64. **PROBLEM SOLVING** Plutonium-238 is a material that generates steady heat due to decay and is used in power systems for some spacecraft. The function $y = a(0.5)^{t/x}$ represents the amount *y* of a substance remaining after *t* years, where *a* is the initial amount and *x* is the length of the half-life (in years).

Plutonium-238
Half-life ≈ 88 years

a. A scientist is studying a 3-gram sample. Write a function that represents the amount *y* of plutonium-238 after *t* years.

b. What is the yearly percent decrease of plutonium-238?

c. Graph the function from part (a). Use the graph to estimate the amount remaining after 12 years.

65. **COMPARING FUNCTIONS** The three given functions describe the amount *y* of ibuprofen (in milligrams) in a person's bloodstream *t* hours after taking the dosage.

$$y \approx 800(0.71)^t$$
$$y \approx 800(0.9943)^{60t}$$
$$y \approx 800(0.843)^{2t}$$

a. Show that these expressions are approximately equivalent.

b. Describe the information given by each of the functions.

66. **COMBINING FUNCTIONS** You deposit \$9000 in a savings account that earns 3.6% annual interest compounded monthly. You also save \$40 per month in a safe at home. Write a function $C(t) = b(t) + h(t)$, where $b(t)$ represents the balance of your savings account and $h(t)$ represents the amount in your safe after t years. What does $C(t)$ represent?

67. **NUMBER SENSE** During a flu epidemic, the number of sick people triples every week. What is the growth rate as a percent? Explain your reasoning.

68. **HOW DO YOU SEE IT?** Match each situation with its graph. Explain your reasoning.

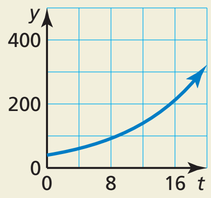
a. A bacterial population doubles each hour.

b. The value of a computer decreases by 18% each year.

c. A deposit earns 11% annual interest compounded yearly.

d. A radioactive element decays 5.5% each year.

A.



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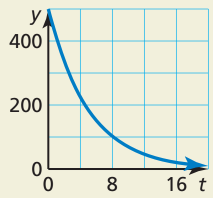
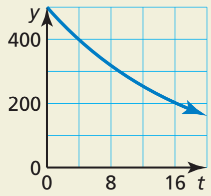
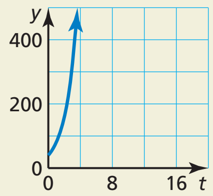


C.



D.

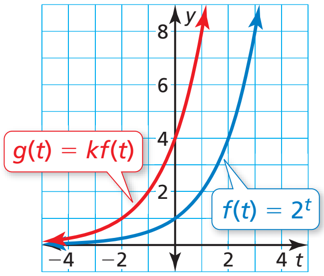


69. **WRITING** Give an example of an equation in the form $y = ab^x$ that does not represent an exponential growth function or an exponential decay function. Explain your reasoning.

70. **THOUGHT PROVOKING** Describe two account options into which you can deposit \$1000 and earn compound interest. Write a function that represents the balance of each account after t years. Which account would you rather use? Explain your reasoning.

71. **MAKING AN ARGUMENT** A store is having a sale on sweaters. On the first day, the prices of the sweaters are reduced by 20%. The prices will be reduced another 20% each day until the sweaters are sold. Your friend says the sweaters will be free on the fifth day. Is your friend correct? Explain.

72. **COMPARING FUNCTIONS** The graphs of f and g are shown.



a. Explain why f is an exponential growth function. Identify the rate of growth.

b. Describe the transformation from the graph of f to the graph of g. Determine the value of k.

c. The graph of g is the same as the graph of $h(t) = f(t + r)$. Use properties of exponents to find the value of r.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Section 1.3)*

73. $8x + 12 = 4x$

74. $5 - t = 7t + 21$

75. $6(r - 2) = 2r + 8$

Find the slope and the y-intercept of the graph of the linear equation. *(Section 3.5)*

76. $y = -6x + 7$

77. $y = \frac{1}{4}x + 7$

78. $3y = 6x - 12$

79. $2y + x = 8$

6.1–6.4 What Did You Learn?

Core Vocabulary

*n*th root of *a*, *p. 300*
radical, *p. 300*
index of a radical, *p. 300*
exponential function, *p. 306*
exponential growth, *p. 314*
exponential growth function, *p. 314*
exponential decay, *p. 315*
exponential decay function, *p. 315*
compound interest, *p. 317*

Core Concepts

Section 6.1

Zero Exponent, *p. 292*
Negative Exponents, *p. 292*
Product of Powers Property, *p. 293*
Quotient of Powers Property, *p. 293*
Power of a Power Property, *p. 293*
Power of a Product Property, *p. 294*
Power of a Quotient Property, *p. 294*

Section 6.2

Real *n*th Roots of *a*, *p. 300*
Rational Exponents, *p. 301*

Section 6.3

Graphing $y = ab^x$ When $b > 1$, *p. 307*
Graphing $y = ab^x$ When $0 < b < 1$, *p. 307*

Section 6.4

Exponential Growth Functions, *p. 314*
Exponential Decay Functions, *p. 315*
Compound Interest, *p. 317*

Mathematical Practices

1. How did you apply what you know to simplify the complicated situation in Exercise 56 on page 297?
2. How can you use previously established results to construct an argument in Exercise 44 on page 304?
3. How is the form of the function you wrote in Exercise 66 on page 322 related to the forms of other types of functions you have learned about in this course?

Study Skills

Analyzing Your Errors

Misreading Directions

- **What Happens:** You incorrectly read or do not understand directions.
- **How to Avoid This Error:** Read the instructions for exercises at least twice and make sure you understand what they mean. Make this a habit and use it when taking tests.

6.1–6.4 Quiz

Simplify the expression. Write your answer using only positive exponents. *(Section 6.1)*

1. $3^2 \cdot 3^4$

2. $(k^4)^{-3}$

3. $\left(\frac{4r^2}{3s^5}\right)^3$

4. $\left(\frac{2x^0}{4x^{-2}y^4}\right)^2$

Evaluate the expression. *(Section 6.2)*

5. $\sqrt[3]{27}$

6. $\left(\frac{1}{16}\right)^{1/4}$

7. $512^{2/3}$

8. $(\sqrt{4})^5$

Graph the function. Describe the domain and range. *(Section 6.3)*

9. $y = 5^x$

10. $y = -2\left(\frac{1}{6}\right)^x$

11. $y = 6(2)^{x-4} - 1$

Determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain. *(Section 6.4)*

12.

x	0	1	2	3
y	7	21	63	189

13.

x	1	2	3	4
y	14,641	1331	121	11

Determine whether the function represents *exponential growth* or *exponential decay*. Identify the percent rate of change. *(Section 6.4)*

14. $y = 3(1.88)^t$

15. $f(t) = \frac{1}{3}(1.26)^t$

16. $f(t) = 80\left(\frac{3}{5}\right)^t$

17. The table shows several units of mass. *(Section 6.1)*

Unit of mass	kilogram	hectogram	dekagram	decigram	centigram	milligram	microgram	nanogram
Mass (in grams)	10^3	10^2	10^1	10^{-1}	10^{-2}	10^{-3}	10^{-6}	10^{-9}

a. How many times larger is a kilogram than a nanogram? Write your answer using only positive exponents.

b. How many times smaller is a milligram than a hectogram? Write your answer using only positive exponents.

c. Which is greater, 10,000 milligrams or 1000 decigrams? Explain your reasoning.

18. You store blankets in a cedar chest. What is the volume of the cedar chest? *(Section 6.2)*

19. The function $f(t) = 5(4)^t$ represents the number of frogs in a pond after t years. *(Section 6.3 and Section 6.4)*

a. Does the function represent *exponential growth* or *exponential decay*? Explain.

b. Graph the function. Describe the domain and range.

c. What is the yearly percent change? the approximate monthly percent change?

d. How many frogs are in the pond after 4 years?

6.5 Solving Exponential Equations

Essential Question

How can you solve an exponential equation graphically?

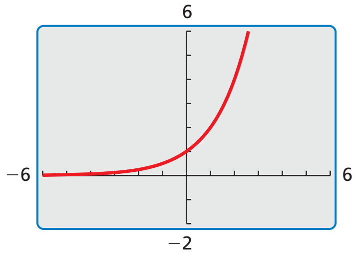
EXPLORATION 1 Solving an Exponential Equation Graphically

Work with a partner. Use a graphing calculator to solve the exponential equation $2.5^{x-3} = 6.25$ graphically. Describe your process and explain how you determined the solution.

EXPLORATION 2 The Number of Solutions of an Exponential Equation

Work with a partner.

a. Use a graphing calculator to graph the equation $y = 2^x$.



b. In the same viewing window, graph a linear equation (if possible) that does not intersect the graph of $y = 2^x$.

c. In the same viewing window, graph a linear equation (if possible) that intersects the graph of $y = 2^x$ in more than one point.

d. Is it possible for an exponential equation to have no solution? more than one solution? Explain your reasoning.

USING APPROPRIATE TOOLS

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

EXPLORATION 3 Solving Exponential Equations Graphically

Work with a partner. Use a graphing calculator to solve each equation.

a. $2^x = \frac{1}{2}$

b. $2^{x+1} = 0$

c. $2^x = \sqrt{2}$

d. $3^x = 9$

e. $3^{x-1} = 0$

f. $4^{2x} = 2$

g. $2^{x/2} = \frac{1}{4}$

h. $3^{x+2} = \frac{1}{9}$

i. $2^{x-2} = \frac{3}{2}x - 2$

Communicate Your Answer

4. How can you solve an exponential equation graphically?

5. A population of 30 mice is expected to double each year. The number p of mice in the population each year is given by $p = 30(2^n)$. In how many years will there be 960 mice in the population?

6.5 Lesson

Core Vocabulary

exponential equation, *p. 326*

What You Will Learn

- Solve exponential equations with the same base.
- Solve exponential equations with unlike bases.
- Solve exponential equations by graphing.

Solving Exponential Equations with the Same Base

Exponential equations are equations in which variable expressions occur as exponents.

Core Concept

Property of Equality for Exponential Equations

Words Two powers with the *same positive base b*, where $b \neq 1$, are equal if and only if their exponents are equal.

Numbers If $2^x = 2^5$, then $x = 5$. If $x = 5$, then $2^x = 2^5$.

Algebra If $b > 0$ and $b \neq 1$, then $b^x = b^y$ if and only if $x = y$.

EXAMPLE 1 Solving Exponential Equations with the Same Base

Solve each equation.

a. $3^{x+1} = 3^5$ **b.** $6 = 6^{2x-3}$ **c.** $10^{3x} = 10^{2x+3}$

SOLUTION

a.

$$\begin{aligned} 3^{x+1} &= 3^5 && \text{Write the equation.} \\ x + 1 &= 5 && \text{Equate the exponents.} \\ \underline{-1} & \quad \underline{-1} && \text{Subtract 1 from each side.} \\ x &= 4 && \text{Simplify.} \end{aligned}$$

b.

$$\begin{aligned} 6 &= 6^{2x-3} && \text{Write the equation.} \\ 1 &= 2x - 3 && \text{Equate the exponents.} \\ \underline{+3} & \quad \underline{+3} && \text{Add 3 to each side.} \\ 4 &= 2x && \text{Simplify.} \\ \frac{4}{2} &= \frac{2x}{2} && \text{Divide each side by 2.} \\ 2 &= x && \text{Simplify.} \end{aligned}$$

Check

$$6 = 6^{2x-3}$$

$$6 \overset{?}{=} 6^{2(2)-3}$$

$$6 = 6 \checkmark$$

c.

$$\begin{aligned} 10^{3x} &= 10^{2x+3} && \text{Write the equation.} \\ 3x &= 2x + 3 && \text{Equate the exponents.} \\ \underline{-2x} & \quad \underline{-2x} && \text{Subtract } 2x \text{ from each side.} \\ x &= 3 && \text{Simplify.} \end{aligned}$$

Monitoring Progress

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Solve the equation. Check your solution.

1. $2^{2x} = 2^6$ **2.** $5^{2x} = 5^{x+1}$ **3.** $7^{3x+5} = 7^{x+1}$

Solving Exponential Equations with Unlike Bases

To solve some exponential equations, you must first rewrite each side of the equation using the same base.

EXAMPLE 2 Solving Exponential Equations with Unlike Bases

Solve (a) $5^x = 125$, (b) $4^x = 2^{x-3}$, and (c) $9^{x+2} = 27^x$.

SOLUTION

a.	$5^x = 125$	Write the equation.
	$5^x = 5^3$	Rewrite 125 as 5^3.
	$x = 3$	Equate the exponents.
b.	$4^x = 2^{x-3}$	Write the equation.
	$(2^2)^x = 2^{x-3}$	Rewrite 4 as 2^2.
	$2^{2x} = 2^{x-3}$	Power of a Power Property
	$2x = x - 3$	Equate the exponents.
	$x = -3$	Solve for x.
c.	$9^{x+2} = 27^x$	Write the equation.
	$(3^2)^{x+2} = (3^3)^x$	Rewrite 9 as 3^2 and 27 as 3^3.
	$3^{2x+4} = 3^{3x}$	Power of a Power Property
	$2x + 4 = 3x$	Equate the exponents.
	$4 = x$	Solve for x.

Check

$4^x = 2^{x-3}$

$4^{-3} \overset{?}{=} 2^{-3-3}$

$\frac{1}{64} = \frac{1}{64}$ ✓

Check

$9^{x+2} = 27^x$

$9^{4+2} \overset{?}{=} 27^4$

$531{,}441 = 531{,}441$ ✓

EXAMPLE 3 Solving Exponential Equations When $0 < b < 1$

Solve (a) $\left(\frac{1}{2}\right)^x = 4$ and (b) $4^{x+1} = \frac{1}{64}$.

SOLUTION

a.	$\left(\frac{1}{2}\right)^x = 4$	Write the equation.
	$(2^{-1})^x = 2^2$	Rewrite $\frac{1}{2}$ as 2^{-1} and 4 as 2^2.
	$2^{-x} = 2^2$	Power of a Power Property
	$-x = 2$	Equate the exponents.
	$x = -2$	Solve for x.
b.	$4^{x+1} = \frac{1}{64}$	Write the equation.
	$4^{x+1} = \frac{1}{4^3}$	Rewrite 64 as 4^3.
	$4^{x+1} = 4^{-3}$	Definition of negative exponent
	$x + 1 = -3$	Equate the exponents.
	$x = -4$	Solve for x.

Check

$4^{x+1} = \frac{1}{64}$

$4^{-4+1} \overset{?}{=} \frac{1}{64}$

$\frac{1}{64} = \frac{1}{64}$ ✓

Monitoring Progress

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Solve the equation. Check your solution.

4. $4^x = 256$

5. $9^{2x} = 3^{x-6}$

6. $4^{3x} = 8^{x+1}$

7. $\left(\frac{1}{3}\right)^{x-1} = 27$

Solving Exponential Equations by Graphing

Sometimes, it is impossible to rewrite each side of an exponential equation using the same base. You can solve these types of equations by graphing each side and finding the point(s) of intersection. Exponential equations can have no solution, one solution, or more than one solution depending on the number of points of intersection.

EXAMPLE 4 Solving Exponential Equations by Graphing

Use a graphing calculator to solve (a) $\left(\frac{1}{2}\right)^{x-1} = 7$ and (b) $3^{x+2} = x + 1$.

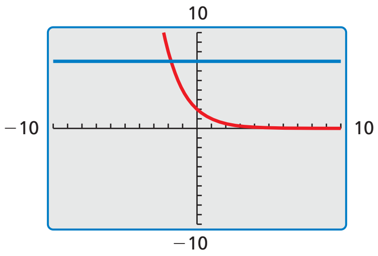
SOLUTION

a. Step 1 Write a system of equations using each side of the equation.

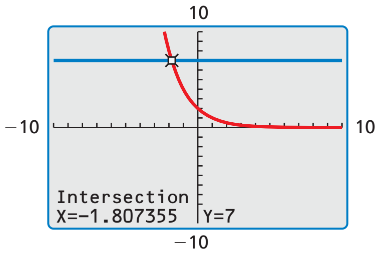
$y = \left(\frac{1}{2}\right)^{x-1}$ Equation 1

$y = 7$ Equation 2

Step 2 Enter the equations into a calculator. Then graph the equations in a viewing window that shows where the graphs could intersect.



Step 3 Use the *intersect* feature to find the point of intersection. The graphs intersect at about (−1.81, 7).



```
Intersection
X=-1.807355  Y=7
```

▶ So, the solution is $x \approx -1.81$.

Check

$\left(\frac{1}{2}\right)^{x-1} = 7$

$\left(\frac{1}{2}\right)^{-1.81-1} \overset{?}{=} 7$

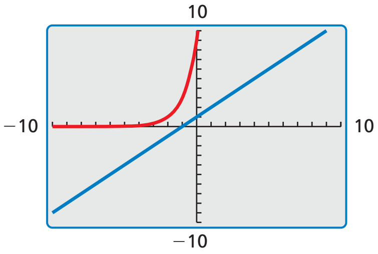
$7.01 \approx 7$ ✓

b. Step 1 Write a system of equations using each side of the equation.

$y = 3^{x+2}$ Equation 1

$y = x + 1$ Equation 2

Step 2 Enter the equations into a calculator. Then graph the equations in a viewing window that shows where the graphs could intersect.



▶ The graphs do not intersect. So, the equation has no solution.

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Use a graphing calculator to solve the equation.

8. $2^x = 1.8$

9. $4^{x-3} = x + 2$

10. $\left(\frac{1}{4}\right)^x = -2x - 3$

6.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Describe how to solve an exponential equation with unlike bases.

2. **WHICH ONE DOESN'T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

$2^x = 4^{x+6}$

$5^{3x+8} = 5^{2x}$

$3^4 = x + 4^2$

$2^{x-7} = 2^7$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, solve the equation. Check your solution. *(See Examples 1 and 2.)*

3. $4^{5x} = 4^{10}$
4. $7^{x-4} = 7^8$
5. $3^{9x} = 3^{7x+8}$
6. $2^{4x} = 2^{x+9}$
7. $2^x = 64$
8. $3^x = 243$
9. $7^{x-5} = 49^x$
10. $216^x = 6^{x+10}$
11. $64^{2x+4} = 16^{5x}$
12. $27^x = 9^{x-2}$

In Exercises 13–18, solve the equation. Check your solution. *(See Example 3.)*

13. $\left(\frac{1}{5}\right)^x = 125$
14. $\left(\frac{1}{4}\right)^x = 256$
15. $\frac{1}{128} = 2^{5x+3}$
16. $3^{4x-9} = \frac{1}{243}$
17. $36^{-3x+3} = \left(\frac{1}{216}\right)^{x+1}$
18. $\left(\frac{1}{27}\right)^{4-x} = 9^{2x-1}$

ERROR ANALYSIS **In Exercises 19 and 20, describe and correct the error in solving the exponential equation.**

19.

$$5^{3x+2} = 25^{x-8}$$
$$3x + 2 = x - 8$$
$$x = -5$$

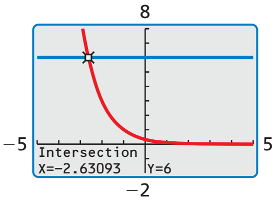
20.

$$\left(\frac{1}{8}\right)^{5x} = 32^{x+8}$$
$$(2^3)^{5x} = (2^5)^{x+8}$$
$$2^{15x} = 2^{5x+40}$$
$$15x = 5x + 40$$
$$x = 4$$

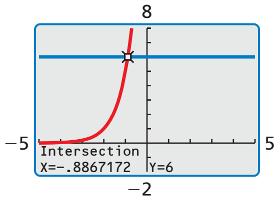
In Exercises 21–24, match the equation with the graph that can be used to solve it. Then solve the equation.

21. $2^x = 6$
22. $4^{2x-5} = 6$
23. $5^{x+2} = 6$
24. $3^{-x-1} = 6$

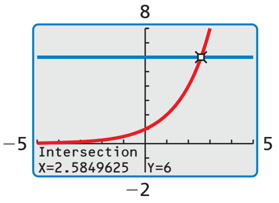
A.



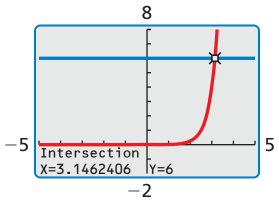
B.



C.



D.



In Exercises 25–36, use a graphing calculator to solve the equation. *(See Example 4.)*

25. $6^{x+2} = 12$
26. $5^{x-4} = 8$
27. $\left(\frac{1}{2}\right)^{7x+1} = -9$
28. $\left(\frac{1}{3}\right)^{x+3} = 10$
29. $2^{x+6} = 2x + 15$
30. $3x - 2 = 5^{x-1}$
31. $\frac{1}{2}x - 1 = \left(\frac{1}{3}\right)^{2x-1}$
32. $2^{-x+1} = -\frac{3}{4}x + 3$
33. $5^x = -4^{-x+4}$
34. $7^{x-2} = 2^{-x}$
35. $2^{-x-3} = 3^{x+1}$
36. $5^{-2x+3} = -6^{x+5}$

In Exercises 37–40, solve the equation by using the Property of Equality for Exponential Equations.

37. $30 \cdot 5^{x+3} = 150$

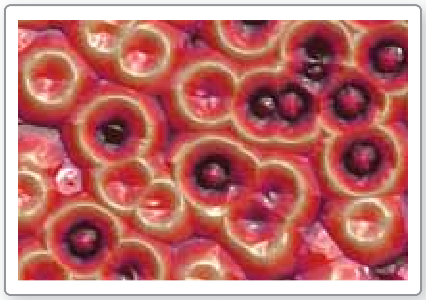
38. $12 \cdot 2^{x-7} = 24$

39. $4(3^{-2x-4}) = 36$

40. $2(4^{2x+1}) = 128$

41. MODELING WITH MATHEMATICS You scan a photo into a computer at four times its original size. You continue to increase its size repeatedly by 100% using the computer. The new size of the photo y in comparison to its original size after x enlargements on the computer is represented by $y = 2^{x+2}$. How many times must the photo be enlarged on the computer so the new photo is 32 times the original size?

42. MODELING WITH MATHEMATICS A bacterial culture quadruples in size every hour. You begin observing the number of bacteria 3 hours after the culture is prepared. The amount y of bacteria x hours after the culture is prepared is represented by $y = 192(4^{x-3})$. When will there be 200,000 bacteria?



In Exercises 43–46, solve the equation.

43. $3^{3x+6} = 27^{x+2}$

44. $3^{4x+3} = 81^{x}$

45. $4^{x+3} = 2^{2(x+1)}$

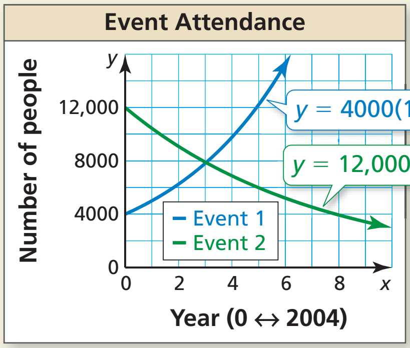
46. $5^{8(x-1)} = 625^{2x-2}$

47. NUMBER SENSE Explain how you can use mental math to solve the equation $8^{x-4} = 1$.

48. PROBLEM SOLVING There are a total of 128 teams at the start of a citywide 3-on-3 basketball tournament. Half the teams are eliminated after each round. Write and solve an exponential equation to determine after which round there are 16 teams left.

49. PROBLEM SOLVING You deposit \$500 in a savings account that earns 6% annual interest compounded yearly. Write and solve an exponential equation to determine when the balance of the account will be \$800.

50. HOW DO YOU SEE IT? The graph shows the annual attendance at two different events. Each event began in 2004.



a. Estimate when the events will have about the same attendance.

b. Explain how you can verify your answer in part (a).

51. REASONING Explain why the Property of Equality for Exponential Equations does not work when $b = 1$. Give an example to justify your answer.

52. THOUGHT PROVOKING Is it possible for an exponential equation to have two different solutions? If not, explain your reasoning. If so, give an example.

USING STRUCTURE In Exercises 53–58, solve the equation.

53. $8^{x-2} = \sqrt{8}$

54. $\sqrt{5} = 5^{x+4}$

55. $\left(\sqrt[5]{7}\right)^{x} = 7^{2x+3}$

56. $12^{2x-1} = \left(\sqrt[4]{12}\right)^{x}$

57. $\left(\sqrt[3]{6}\right)^{2x} = \left(\sqrt{6}\right)^{x+6}$

58. $\left(\sqrt[5]{3}\right)^{5x-10} = \left(\sqrt[8]{3}\right)^{4x}$

59. MAKING AN ARGUMENT Consider the equation $\left(\frac{1}{a}\right)^{x} = b$, where $a > 1$ and $b > 1$. Your friend says the value of x will always be negative. Is your friend correct? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Determine whether the sequence is arithmetic. If so, find the common difference. *(Section 4.6)*

60. −20, −26, −32, −38, . . .

61. 9, 18, 36, 72, . . .

62. −5, −8, −12, −17, . . .

63. 10, 20, 30, 40, . . .

6.6 Geometric Sequences

Essential Question

How can you use a geometric sequence to describe a pattern?

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**.

EXPLORATION 1 Describing Calculator Patterns

Work with a partner. Enter the keystrokes on a calculator and record the results in the table. Describe the pattern.

a. Step 1 2 =
Step 2 × 2 =
Step 3 × 2 =
Step 4 × 2 =
Step 5 × 2 =

b. Step 1 6 4 =
Step 2 × . 5 =
Step 3 × . 5 =
Step 4 × . 5 =
Step 5 × . 5 =

Step	1	2	3	4	5
Calculator display					

Step	1	2	3	4	5
Calculator display					

c. Use a calculator to make your own sequence. Start with any number and multiply by 3 each time. Record your results in the table.

Step	1	2	3	4	5
Calculator display					

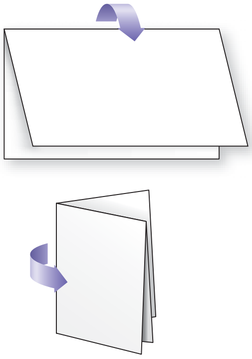
d. Part (a) involves a geometric sequence with a common ratio of 2. What is the common ratio in part (b)? part (c)?

LOOKING FOR REGULARITY IN REPEATED REASONING

To be proficient in math, you need to notice when calculations are repeated and look both for general methods and for shortcuts.

EXPLORATION 2 Folding a Sheet of Paper

Work with a partner. A sheet of paper is about 0.1 millimeter thick.



a. How thick will it be when you fold it in half once? twice? three times?

b. What is the greatest number of times you can fold a piece of paper in half? How thick is the result?

c. Do you agree with the statement below? Explain your reasoning.

"If it were possible to fold the paper in half 15 times, it would be taller than you."

Communicate Your Answer

3. How can you use a geometric sequence to describe a pattern?

4. Give an example of a geometric sequence from real life other than paper folding.

6.6 Lesson

Core Vocabulary

geometric sequence, *p. 332*
common ratio, *p. 332*

Previous
arithmetic sequence
common difference

What You Will Learn

- Identify geometric sequences.
- Extend and graph geometric sequences.
- Write geometric sequences as functions.

Identifying Geometric Sequences

Core Concept

Geometric Sequence

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**. Each term is found by multiplying the previous term by the common ratio.

$1, \quad 5, \quad 25, \quad 125, \ldots$ Terms of a geometric sequence

$\times 5 \quad \times 5 \quad \times 5$ ← common ratio

EXAMPLE 1 Identifying Geometric Sequences

Decide whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

a. 120, 60, 30, 15, . . . **b.** 2, 6, 11, 17, . . .

SOLUTION

a. Find the ratio between each pair of consecutive terms.

$120 \quad 60 \quad 30 \quad 15$

$\frac{60}{120} = \frac{1}{2} \quad \frac{30}{60} = \frac{1}{2} \quad \frac{15}{30} = \frac{1}{2}$ The ratios are the same. The common ratio is $\frac{1}{2}$.

- So, the sequence is geometric.

b. Find the ratio between each pair of consecutive terms.

$2 \quad 6 \quad 11 \quad 17$

$\frac{6}{2} = 3 \quad \frac{11}{6} = 1\frac{5}{6} \quad \frac{17}{11} = 1\frac{6}{11}$ There is no common ratio, so the sequence is *not* geometric.

Find the difference between each pair of consecutive terms.

$2 \quad 6 \quad 11 \quad 17$

$6 - 2 = 4 \quad 11 - 6 = 5 \quad 17 - 11 = 6$ There is no common difference, so the sequence is *not* arithmetic.

- So, the sequence is *neither* geometric nor arithmetic.

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Decide whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

1. 5, 1, −3, −7, . . . **2.** 1024, 128, 16, 2, . . . **3.** 2, 6, 10, 16, . . .

Extending and Graphing Geometric Sequences

EXAMPLE 2 Extending Geometric Sequences

Write the next three terms of each geometric sequence.

a. 3, 6, 12, 24, . . .

b. 64, −16, 4, −1, . . .

SOLUTION

Use tables to organize the terms and extend each sequence.

a.

Position	1	2	3	4	5	6	7
Term	3	6	12	24	48	96	192

× 2 × 2 × 2 × 2 × 2 × 2

Each term is twice the previous term. So, the common ratio is 2.

Multiply a term by 2 to find the next term.

▶ The next three terms are 48, 96, and 192.

b.

Position	1	2	3	4	5	6	7
Term	64	−16	4	−1	$\frac{1}{4}$	$-\frac{1}{16}$	$\frac{1}{64}$

$\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$ $\times\left(-\frac{1}{4}\right)$

Multiply a term by $-\frac{1}{4}$ to find the next term.

▶ The next three terms are $\frac{1}{4}$, $-\frac{1}{16}$, and $\frac{1}{64}$.

LOOKING FOR STRUCTURE

When the terms of a geometric sequence alternate between positive and negative terms, or vice versa, the common ratio is negative.

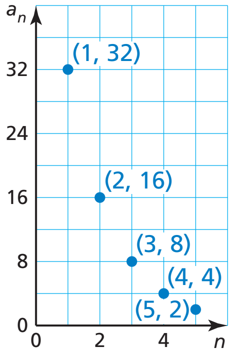
EXAMPLE 3 Graphing a Geometric Sequence

Graph the geometric sequence 32, 16, 8, 4, 2, ... What do you notice?

SOLUTION

Make a table. Then plot the ordered pairs (n, a_n).

Position, n	1	2	3	4	5
Term, a_n	32	16	8	4	2



▶ The points appear to lie on an exponential curve.

STUDY TIP

The points of any geometric sequence with a *positive* common ratio lie on an exponential curve.

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Write the next three terms of the geometric sequence. Then graph the sequence.

4. 1, 3, 9, 27, . . .

5. 2500, 500, 100, 20, . . .

6. 80, −40, 20, −10, . . .

7. −2, 4, −8, 16, . . .

Writing Geometric Sequences as Functions

Because consecutive terms of a geometric sequence have a common ratio, you can use the first term a_1 and the common ratio r to write an exponential function that describes a geometric sequence. Let $a_1 = 1$ and $r = 5$.

Position, n	Term, a_n	Written using a_1 and r	Numbers
1	first term, a_1	a_1	1
2	second term, a_2	$a_1 r$	$1 \cdot 5 = 5$
3	third term, a_3	$a_1 r^2$	$1 \cdot 5^2 = 25$
4	fourth term, a_4	$a_1 r^3$	$1 \cdot 5^3 = 125$
⋮	⋮	⋮	⋮
n	nth term, a_n	$a_1 r^{n-1}$	$1 \cdot 5^{n-1}$

STUDY TIP

Notice that the equation $a_n = a_1 r^{n-1}$ is of the form $y = ab^x$.

Core Concept

Equation for a Geometric Sequence

Let a_n be the nth term of a geometric sequence with first term a_1 and common ratio r. The nth term is given by

$$a_n = a_1 r^{n-1}.$$

EXAMPLE 4 Finding the nth Term of a Geometric Sequence

Write an equation for the nth term of the geometric sequence 2, 12, 72, 432, ... Then find a_{10}.

SOLUTION

The first term is 2, and the common ratio is 6.

$$a_n = a_1 r^{n-1} \qquad \text{Equation for a geometric sequence}$$

$$a_n = 2(6)^{n-1} \qquad \text{Substitute 2 for } a_1 \text{ and 6 for } r.$$

Use the equation to find the 10th term.

$$a_n = 2(6)^{n-1} \qquad \text{Write the equation.}$$

$$a_{10} = 2(6)^{10-1} \qquad \text{Substitute 10 for } n.$$

$$= 20{,}155{,}392 \qquad \text{Simplify.}$$

▶ The 10th term of the geometric sequence is 20,155,392.

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Write an equation for the nth term of the geometric sequence. Then find a_7.

8. 1, −5, 25, −125, . . .

9. 13, 26, 52, 104, . . .

10. 432, 72, 12, 2, . . .

11. 4, 10, 25, 62.5, . . .

You can rewrite the equation for a geometric sequence with first term a_1 and common ratio r in function notation by replacing a_n with $f(n)$.

$$f(n) = a_1 r^{n-1}$$

The domain of the function is the set of positive integers.

EXAMPLE 5 Modeling with Mathematics

Clicking the *zoom-out* button on a mapping website doubles the side length of the square map. After how many clicks on the *zoom-out* button is the side length of the map 640 miles?

Zoom-out clicks	1	2	3
Map side length (miles)	5	10	20

SOLUTION

1. **Understand the Problem** You know that the side length of the square map doubles after each click on the *zoom-out* button. So, the side lengths of the map represent the terms of a geometric sequence. You need to find the number of clicks it takes for the side length of the map to be 640 miles.

2. **Make a Plan** Begin by writing a function f for the nth term of the geometric sequence. Then find the value of n for which $f(n) = 640$.

3. **Solve the Problem** The first term is 5, and the common ratio is 2.

$f(n) = a_1 r^{n-1}$ Function for a geometric sequence

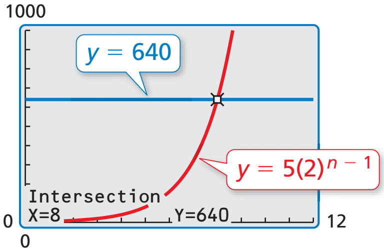
$f(n) = 5(2)^{n-1}$ Substitute 5 for a_1 and 2 for r.

The function $f(n) = 5(2)^{n-1}$ represents the geometric sequence. Use this function to find the value of n for which $f(n) = 640$. So, use the equation $640 = 5(2)^{n-1}$ to write a system of equations.

$y = 5(2)^{n-1}$ Equation 1

$y = 640$ Equation 2

Then use a graphing calculator to graph the equations and find the point of intersection. The point of intersection is (8, 640).



▶ So, after eight clicks, the side length of the map is 640 miles.

4. **Look Back** Find the value of n for which $f(n) = 640$ algebraically.

$640 = 5(2)^{n-1}$ Write the equation.

$128 = (2)^{n-1}$ Divide each side by 5.

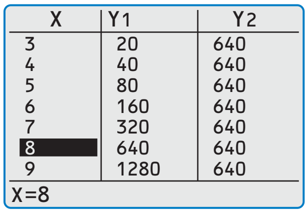
$2^7 = (2)^{n-1}$ Rewrite 128 as 2^7.

$7 = n - 1$ Equate the exponents.

$8 = n$ ✓ Add 1 to each side.

USING APPROPRIATE TOOLS STRATEGICALLY

You can also use the *table* feature of a graphing calculator to find the value of n for which $f(n) = 640$.



X	Y1	Y2
3	20	640
4	40	640
5	80	640
6	160	640
7	320	640
8	640	640
9	1280	640

X=8

Monitoring Progress

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12. **WHAT IF?** After how many clicks on the *zoom-out* button is the side length of the map 2560 miles?

6.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. WRITING Compare the two sequences.

2, 4, 6, 8, 10, . . .   2, 4, 8, 16, 32, . . .

2. CRITICAL THINKING Why do the points of a geometric sequence lie on an exponential curve only when the common ratio is positive?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the common ratio of the geometric sequence.

3. 4, 12, 36, 108, . . .

4. 36, 6, 1, $\frac{1}{6}$, . . .

5. $\frac{3}{8}$, −3, 24, −192, . . .

6. 0.1, 1, 10, 100, . . .

7. 128, 96, 72, 54, . . .

8. −162, 54, −18, 6, . . .

In Exercises 9–14, determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. *(See Example 1.)*

9. −8, 0, 8, 16, . . .

10. −1, 4, −7, 10, . . .

11. 9, 14, 20, 27, . . .

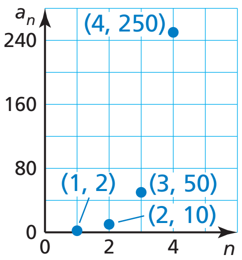
12. $\frac{3}{49}$, $\frac{3}{7}$, 3, 21, . . .

13. 192, 24, 3, $\frac{3}{8}$, . . .

14. −25, −18, −11, −4, . . .

In Exercises 15–18, determine whether the graph represents an *arithmetic sequence*, a *geometric sequence*, or *neither*. Explain your reasoning.

15.



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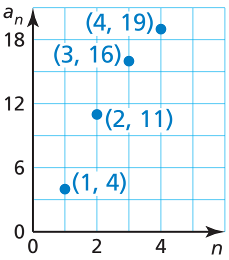

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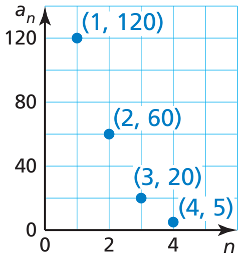

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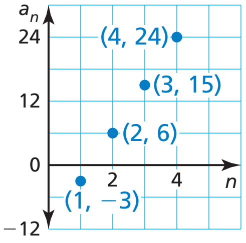


In Exercises 19–24, write the next three terms of the geometric sequence. Then graph the sequence. *(See Examples 2 and 3.)*

19. 5, 20, 80, 320, . . .

20. −3, 12, −48, 192, . . .

21. 81, −27, 9, −3, . . .

22. −375, −75, −15, −3, . . .

23. 32, 8, 2, $\frac{1}{2}$, . . .

24. $\frac{16}{9}$, $\frac{8}{3}$, 4, 6, . . .

In Exercises 25–32, write an equation for the nth term of the geometric sequence. Then find a_6. *(See Example 4.)*

25. 2, 8, 32, 128, . . .

26. 0.6, −3, 15, −75, . . .

27. $-\frac{1}{8}$, $-\frac{1}{4}$, $-\frac{1}{2}$, −1, . . .

28. 0.1, 0.9, 8.1, 72.9, . . .

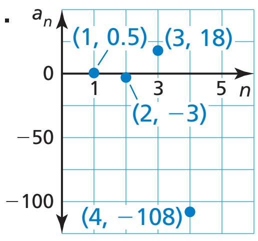
29.

n	1	2	3	4
a_n	7640	764	76.4	7.64

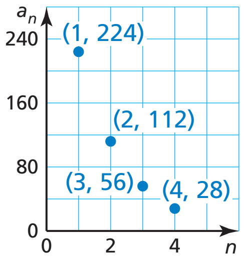
30.

n	1	2	3	4
a_n	−192	48	−12	3

31.

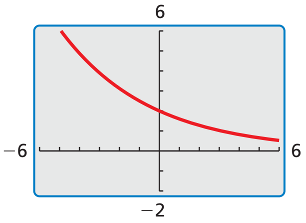


32.



33. PROBLEM SOLVING A badminton tournament begins with 128 teams. After the first round, 64 teams remain. After the second round, 32 teams remain. How many teams remain after the third, fourth, and fifth rounds?

34. **PROBLEM SOLVING** The graphing calculator screen displays an area of 96 square units. After you zoom out once, the area is 384 square units. After you zoom out a second time, the area is 1536 square units. What is the screen area after you zoom out four times?



35. **ERROR ANALYSIS** Describe and correct the error in writing the next three terms of the geometric sequence.

✗ $-8, \xrightarrow[\times(-2)]{} 4, \xrightarrow[\times(-2)]{} -2, \xrightarrow[\times(-2)]{} 1, \ldots$

The next three terms are -2, 4, and -8.

36. **ERROR ANALYSIS** Describe and correct the error in writing an equation for the *n*th term of the geometric sequence.

✗ $-2, -12, -72, -432, \ldots$

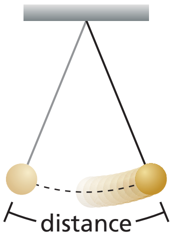
The first term is -2, and the common ratio is -6.

$$a_n = a_1 r^{n-1}$$

$$a_n = -2(-6)^{n-1}$$

37. **MODELING WITH MATHEMATICS** The distance (in millimeters) traveled by a swinging pendulum decreases after each swing, as shown in the table. *(See Example 5.)*

Swing	1	2	3
Distance (in millimeters)	625	500	400



a. Write a function that represents the distance the pendulum swings on its *n*th swing.

b. After how many swings is the distance 256 millimeters?

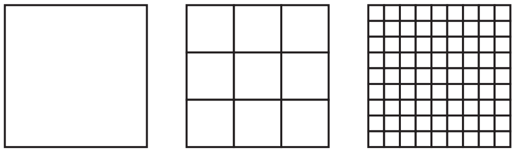
38. **MODELING WITH MATHEMATICS** You start a chain email and send it to six friends. The next day, each of your friends forwards the email to six people. The process continues for a few days.

a. Write a function that represents the number of people who have received the email after *n* days.

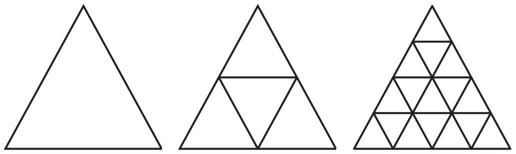
b. After how many days will 1296 people have received the email?

MATHEMATICAL CONNECTIONS **In Exercises 39 and 40, (a) write a function that represents the sequence of figures and (b) describe the 10th figure in the sequence.**

39.



40.



41. **REASONING** Write a sequence that represents the number of teams that have been eliminated after *n* rounds of the badminton tournament in Exercise 33. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

42. **REASONING** Write a sequence that represents the perimeter of the graphing calculator screen in Exercise 34 after you zoom out *n* times. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

43. **WRITING** Compare the graphs of arithmetic sequences to the graphs of geometric sequences.

44. **MAKING AN ARGUMENT** You are given two consecutive terms of a sequence.

$\ldots, -8, 0, \ldots$

Your friend says that the sequence is not geometric. A classmate says that is impossible to know given only two terms. Who is correct? Explain.

45. **CRITICAL THINKING** Is the sequence shown an *arithmetic* sequence? a *geometric* sequence? Explain your reasoning.

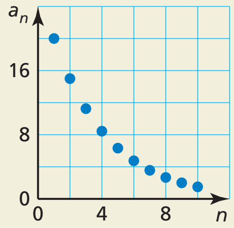
3, 3, 3, 3, . . .

46. **HOW DO YOU SEE IT?** Without performing any calculations, match each equation with its graph. Explain your reasoning.

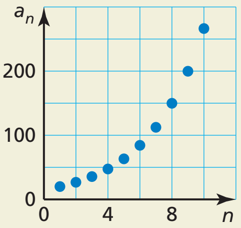
a. $a_n = 20\left(\frac{4}{3}\right)^{n-1}$

b. $a_n = 20\left(\frac{3}{4}\right)^{n-1}$

A.



B.



47. **REASONING** What is the 9th term of the geometric sequence where $a_3 = 81$ and $r = 3$?

48. **OPEN-ENDED** Write a sequence that has a pattern but is not arithmetic or geometric. Describe the pattern.

49. **ATTENDING TO PRECISION** Are the terms of a geometric sequence independent or dependent? Explain your reasoning.

50. **DRAWING CONCLUSIONS** A college student makes a deal with her parents to live at home instead of living on campus. She will pay her parents $0.01 for the first day of the month, $0.02 for the second day, $0.04 for the third day, and so on.

a. Write an equation that represents the *n*th term of the geometric sequence.

b. What will she pay on the 25th day?

c. Did the student make a good choice or should she have chosen to live on campus? Explain.

51. **REPEATED REASONING** A soup kitchen makes 16 gallons of soup. Each day, a quarter of the soup is served and the rest is saved for the next day.

a. Write the first five terms of the sequence of the number of fluid ounces of soup left each day.

b. Write an equation that represents the *n*th term of the sequence.

c. When is all the soup gone? Explain.

52. **THOUGHT PROVOKING** Find the sum of the terms of the geometric sequence.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{n-1}}, \ldots$$

Explain your reasoning. Write a different infinite geometric sequence that has the same sum.

53. **OPEN-ENDED** Write a geometric sequence in which $a_2 < a_1 < a_3$.

54. **NUMBER SENSE** Write an equation that represents the *n*th term of each geometric sequence shown.

n	1	2	3	4
a_n	2	6	18	54

n	1	2	3	4
b_n	1	5	25	125

a. Do the terms $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots$ form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

b. Do the terms $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Use residuals to determine whether the model is a good fit for the data in the table. Explain. *(Section 4.5)*

55. $y = 3x - 8$

x	0	1	2	3	4	5	6
y	−10	−2	−1	2	1	7	10

56. $y = -5x + 1$

x	−3	−2	−1	0	1	2	3
y	6	4	6	1	2	−4	−3

6.7 Recursively Defined Sequences

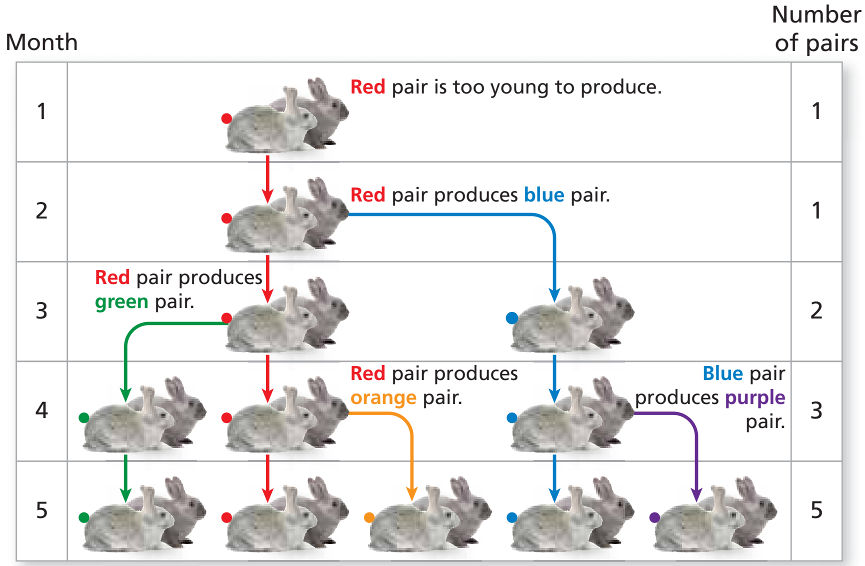
Essential Question

How can you define a sequence recursively?

A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

EXPLORATION 1 Describing a Pattern

Work with a partner. Consider a hypothetical population of rabbits. Start with one breeding pair. After each month, each breeding pair produces another breeding pair. The total number of rabbits each month follows the exponential pattern 2, 4, 8, 16, 32, ... Now suppose that in the first month after each pair is born, the pair is too young to reproduce. Each pair produces another pair after it is 2 months old. Find the total number of pairs in months 6, 7, and 8.



RECOGNIZING PATTERNS

To be proficient in math, you need to look closely to discern a pattern or structure.

EXPLORATION 2 Using a Recursive Equation

Work with a partner. Consider the following recursive equation.

$$a_n = a_{n-1} + a_{n-2}$$

Each term in the sequence is the sum of the two preceding terms.

Copy and complete the table. Compare the results with the sequence of the number of pairs in Exploration 1.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1						

Communicate Your Answer

3. How can you define a sequence recursively?

4. Use the Internet or some other reference to determine the mathematician who first described the sequences in Explorations 1 and 2.

6.7 Lesson

Core Vocabulary

explicit rule, *p. 340*
recursive rule, *p. 340*

Previous
arithmetic sequence
geometric sequence

What You Will Learn

- Write terms of recursively defined sequences.
- Write recursive rules for sequences.
- Translate between recursive rules and explicit rules.
- Write recursive rules for special sequences.

Writing Terms of Recursively Defined Sequences

So far in this book, you have defined arithmetic and geometric sequences *explicitly*. An **explicit rule** gives a_n as a function of the term's position number n in the sequence. For example, an explicit rule for the arithmetic sequence 3, 5, 7, 9, . . . is $a_n = 3 + 2(n - 1)$, or $a_n = 2n + 1$.

Now, you will define arithmetic and geometric sequences *recursively*. A **recursive rule** gives the beginning term(s) of a sequence and a *recursive equation* that tells how a_n is related to one or more preceding terms.

Core Concept

Recursive Equation for an Arithmetic Sequence

$a_n = a_{n-1} + d$, where d is the common difference

Recursive Equation for a Geometric Sequence

$a_n = r \cdot a_{n-1}$, where r is the common ratio

EXAMPLE 1 Writing Terms of Recursively Defined Sequences

Write the first six terms of each sequence. Then graph each sequence.

a. $a_1 = 2, a_n = a_{n-1} + 3$

b. $a_1 = 1, a_n = 3a_{n-1}$

SOLUTION

You are given the first term. Use the recursive equation to find the next five terms.

a. $a_1 = 2$

$a_2 = a_1 + 3 = 2 + 3 = 5$

$a_3 = a_2 + 3 = 5 + 3 = 8$

$a_4 = a_3 + 3 = 8 + 3 = 11$

$a_5 = a_4 + 3 = 11 + 3 = 14$

$a_6 = a_5 + 3 = 14 + 3 = 17$

b. $a_1 = 1$

$a_2 = 3a_1 = 3(1) = 3$

$a_3 = 3a_2 = 3(3) = 9$

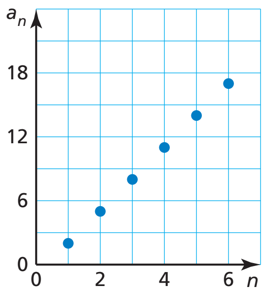
$a_4 = 3a_3 = 3(9) = 27$

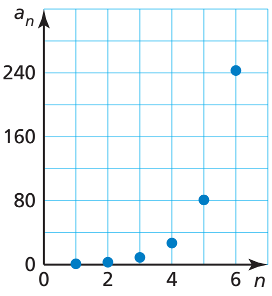
$a_5 = 3a_4 = 3(27) = 81$

$a_6 = 3a_5 = 3(81) = 243$

STUDY TIP

A sequence is a discrete function. So, the points on the graph are not connected.





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Write the first six terms of the sequence. Then graph the sequence.

1. $a_1 = 0, a_n = a_{n-1} - 8$

2. $a_1 = -7.5, a_n = a_{n-1} + 2.5$

3. $a_1 = -36, a_n = \frac{1}{2}a_{n-1}$

4. $a_1 = 0.7, a_n = 10a_{n-1}$

Writing Recursive Rules

EXAMPLE 2 Writing Recursive Rules

Write a recursive rule for each sequence.

a. $-30, -18, -6, 6, 18, \ldots$

b. $500, 100, 20, 4, 0.8, \ldots$

SOLUTION

Use a table to organize the terms and find the pattern.

a.

Position, n	1	2	3	4	5
Term, a_n	-30	-18	-6	6	18

$+12 \quad +12 \quad +12 \quad +12$

The sequence is arithmetic, with first term $a_1 = -30$ and common difference $d = 12$.

$a_n = a_{n-1} + d$ Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + 12$ Substitute 12 for *d*.

▶ So, a recursive rule for the sequence is $a_1 = -30, a_n = a_{n-1} + 12$.

COMMON ERROR

When writing a recursive rule for a sequence, you need to write both the beginning term(s) and the recursive equation.

b.

Position, n	1	2	3	4	5
Term, a_n	500	100	20	4	0.8

$\times \frac{1}{5} \quad \times \frac{1}{5} \quad \times \frac{1}{5} \quad \times \frac{1}{5}$

The sequence is geometric, with first term $a_1 = 500$ and common ratio $r = \frac{1}{5}$.

$a_n = r \cdot a_{n-1}$ Recursive equation for a geometric sequence

$a_n = \frac{1}{5}a_{n-1}$ Substitute $\frac{1}{5}$ for *r*.

▶ So, a recursive rule for the sequence is $a_1 = 500, a_n = \frac{1}{5}a_{n-1}$.

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Write a recursive rule for the sequence.

5. $8, 3, -2, -7, -12, \ldots$

6. $1.3, 2.6, 3.9, 5.2, 6.5, \ldots$

7. $4, 20, 100, 500, 2500, \ldots$

8. $128, -32, 8, -2, 0.5, \ldots$

9. Write a recursive rule for the height of the sunflower over time.

Translating between Recursive and Explicit Rules

EXAMPLE 3 Translating from Recursive Rules to Explicit Rules

Write an explicit rule for each recursive rule.

a. $a_1 = 25, a_n = a_{n-1} - 10$

b. $a_1 = 19.6, a_n = -0.5a_{n-1}$

SOLUTION

a. The recursive rule represents an arithmetic sequence, with first term $a_1 = 25$ and common difference $d = -10$.

$a_n = a_1 + (n - 1)d$ Explicit rule for an arithmetic sequence

$a_n = 25 + (n - 1)(-10)$ Substitute 25 for a_1 and −10 for d.

$a_n = -10n + 35$ Simplify.

▶ An explicit rule for the sequence is $a_n = -10n + 35$.

b. The recursive rule represents a geometric sequence, with first term $a_1 = 19.6$ and common ratio $r = -0.5$.

$a_n = a_1r^{n-1}$ Explicit rule for a geometric sequence

$a_n = 19.6(-0.5)^{n-1}$ Substitute 19.6 for a_1 and −0.5 for r.

▶ An explicit rule for the sequence is $a_n = 19.6(-0.5)^{n-1}$.

EXAMPLE 4 Translating from Explicit Rules to Recursive Rules

Write a recursive rule for each explicit rule.

a. $a_n = -2n + 3$

b. $a_n = -3(2)^{n-1}$

SOLUTION

a. The explicit rule represents an arithmetic sequence, with first term $a_1 = -2(1) + 3 = 1$ and common difference $d = -2$.

$a_n = a_{n-1} + d$ Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + (-2)$ Substitute −2 for d.

▶ So, a recursive rule for the sequence is $a_1 = 1, a_n = a_{n-1} - 2$.

b. The explicit rule represents a geometric sequence, with first term $a_1 = -3$ and common ratio $r = 2$.

$a_n = r \cdot a_{n-1}$ Recursive equation for a geometric sequence

$a_n = 2a_{n-1}$ Substitute 2 for r.

▶ So, a recursive rule for the sequence is $a_1 = -3, a_n = 2a_{n-1}$.

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Write an explicit rule for the recursive rule.

10. $a_1 = -45, a_n = a_{n-1} + 20$

11. $a_1 = 13, a_n = -3a_{n-1}$

Write a recursive rule for the explicit rule.

12. $a_n = -n + 1$

13. $a_n = -2.5(4)^{n-1}$

Writing Recursive Rules for Special Sequences

You can write recursive rules for sequences that are neither arithmetic nor geometric. One way is to look for patterns in the sums of consecutive terms.

EXAMPLE 5 Writing Recursive Rules for Other Sequences

The sequence in Example 5 is called the *Fibonacci sequence*. This pattern is naturally occurring in many objects, such as flowers.

Use the sequence shown.

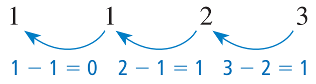
$$1, 1, 2, 3, 5, 8, \ldots$$

a. Write a recursive rule for the sequence.

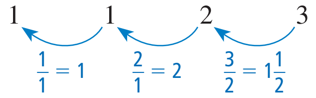
b. Write the next three terms of the sequence.

SOLUTION

a. Find the difference and ratio between each pair of consecutive terms.



There is no common difference, so the sequence is *not* arithmetic.



There is no common ratio, so the sequence is *not* geometric.

Find the sum of each pair of consecutive terms.

$a_1 + a_2 = 1 + 1 = 2$ 2 is the third term.

$a_2 + a_3 = 1 + 2 = 3$ 3 is the fourth term.

$a_3 + a_4 = 2 + 3 = 5$ 5 is the fifth term.

$a_4 + a_5 = 3 + 5 = 8$ 8 is the sixth term.

Beginning with the third term, each term is the sum of the two previous terms. A recursive equation for the sequence is $a_n = a_{n-2} + a_{n-1}$.

▶ So, a recursive rule for the sequence is $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$.

b. Use the recursive equation $a_n = a_{n-2} + a_{n-1}$ to find the next three terms.

$$\begin{aligned} a_7 &= a_5 + a_6 \\ &= 5 + 8 \\ &= 13 \end{aligned} \qquad \begin{aligned} a_8 &= a_6 + a_7 \\ &= 8 + 13 \\ &= 21 \end{aligned} \qquad \begin{aligned} a_9 &= a_7 + a_8 \\ &= 13 + 21 \\ &= 34 \end{aligned}$$

▶ The next three terms are 13, 21, and 34.

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Write a recursive rule for the sequence. Then write the next three terms of the sequence.

14. 5, 6, 11, 17, 28, . . .

15. −3, −4, −7, −11, −18, . . .

16. 1, 1, 0, −1, −1, 0, 1, 1, . . .

17. 4, 3, 1, 2, −1, 3, −4, . . .

6.7 Exercises

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Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A recursive rule gives the beginning term(s) of a sequence and a(n) ____________ that tells how a_n is related to one or more preceding terms.

2. **WHICH ONE DOESN'T BELONG?** Which rule does *not* belong with the other three? Explain your reasoning.

$a_1 = -1, a_n = 5a_{n-1}$

$a_n = 6n - 2$

$a_1 = -3, a_n = a_{n-1} + 1$

$a_1 = 9, a_n = 4a_{n-1}$

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, determine whether the recursive rule represents an *arithmetic sequence* or a *geometric sequence*.

3. $a_1 = 2, a_n = 7a_{n-1}$

4. $a_1 = 18, a_n = a_{n-1} + 1$

5. $a_1 = 5, a_n = a_{n-1} - 4$

6. $a_1 = 3, a_n = -6a_{n-1}$

In Exercises 7–12, write the first six terms of the sequence. Then graph the sequence. *(See Example 1.)*

7. $a_1 = 0, a_n = a_{n-1} + 2$

8. $a_1 = 10, a_n = a_{n-1} - 5$

9. $a_1 = 2, a_n = 3a_{n-1}$

10. $a_1 = 8, a_n = 1.5a_{n-1}$

11. $a_1 = 80, a_n = -\frac{1}{2}a_{n-1}$

12. $a_1 = -7, a_n = -4a_{n-1}$

In Exercises 13–20, write a recursive rule for the sequence. *(See Example 2.)*

13.

n	1	2	3	4
a_n	7	16	25	34

14.

n	1	2	3	4
a_n	8	24	72	216

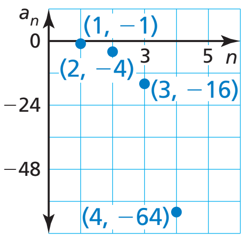
15. 243, 81, 27, 9, 3, . . .

16. 3, 11, 19, 27, 35, . . .

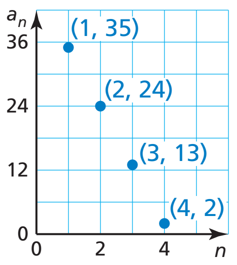
17. 0, −3, −6, −9, −12, . . .

18. 5, −20, 80, −320, 1280, . . .

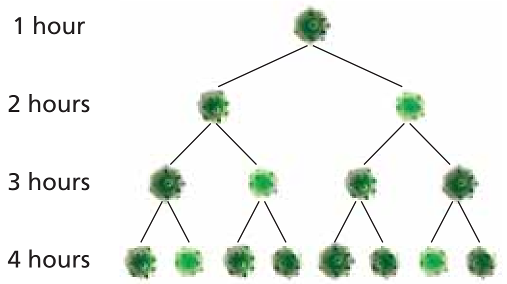
19.



20.



21. **MODELING WITH MATHEMATICS** Write a recursive rule for the number of bacterial cells over time.



22. **MODELING WITH MATHEMATICS** Write a recursive rule for the length of the deer antler over time.

In Exercises 23–28, write an explicit rule for the recursive rule. *(See Example 3.)*

23. $a_1 = -3, a_n = a_{n-1} + 3$

24. $a_1 = 8, a_n = a_{n-1} - 12$

25. $a_1 = 16, a_n = 0.5a_{n-1}$

26. $a_1 = -2, a_n = 9a_{n-1}$

27. $a_1 = 4, a_n = a_{n-1} + 17$

28. $a_1 = 5, a_n = -5a_{n-1}$

In Exercises 29–34, write a recursive rule for the explicit rule. *(See Example 4.)*

29. $a_n = 7(3)^{n-1}$

30. $a_n = -4n + 2$

31. $a_n = 1.5n + 3$

32. $a_n = 6n - 20$

33. $a_n = (-5)^{n-1}$

34. $a_n = -81\left(\frac{2}{3}\right)^{n-1}$

In Exercises 35–38, graph the first four terms of the sequence with the given description. Write a recursive rule and an explicit rule for the sequence.

35. The first term of a sequence is 5. Each term of the sequence is 15 more than the preceding term.

36. The first term of a sequence is 16. Each term of the sequence is half the preceding term.

37. The first term of a sequence is −1. Each term of the sequence is −3 times the preceding term.

38. The first term of a sequence is 19. Each term of the sequence is 13 less than the preceding term.

In Exercises 39–44, write a recursive rule for the sequence. Then write the next two terms of the sequence. *(See Example 5.)*

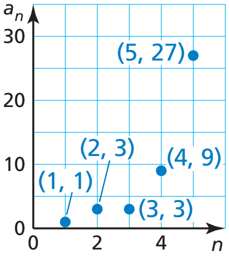
39. 1, 3, 4, 7, 11, . . .

40. 10, 9, 1, 8, −7, 15, . . .

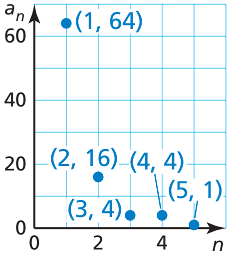
41. 2, 4, 2, −2, −4, −2, . . .

42. 6, 1, 7, 8, 15, 23, . . .

43.



44.



45. **ERROR ANALYSIS** Describe and correct the error in writing an explicit rule for the recursive rule $a_1 = 6$, $a_n = a_{n-1} - 12$.

✗

$a_n = a_1 + (n - 1)d$

$a_n = 6 + (n - 1)(12)$

$a_n = 6 + 12n - 12$

$a_n = -6 + 12n$

46. **ERROR ANALYSIS** Describe and correct the error in writing a recursive rule for the sequence 2, 4, 6, 10, 16, ...

✗

$2, \underset{+2}{\curvearrowright} 4, \underset{+2}{\curvearrowright} 6, \ldots$

The sequence is arithmetic, with first term $a_1 = 2$ and common difference $d = 2$.

$a_n = a_{n-1} + d$

$a_1 = 2, a_n = a_{n-1} + 2$

In Exercises 47–51, the function *f* represents a sequence. Find the 2nd, 5th, and 10th terms of the sequence.

47. $f(1) = 3, f(n) = f(n - 1) + 7$

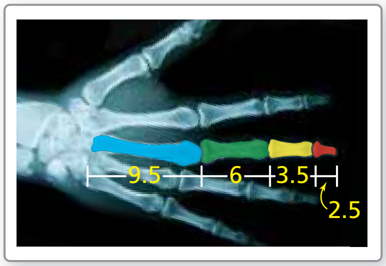
48. $f(1) = -1, f(n) = 6f(n - 1)$

49. $f(1) = 8, f(n) = -f(n - 1)$

50. $f(1) = 4, f(2) = 5, f(n) = f(n - 2) + f(n - 1)$

51. $f(1) = 10, f(2) = 15, f(n) = f(n - 1) - f(n - 2)$

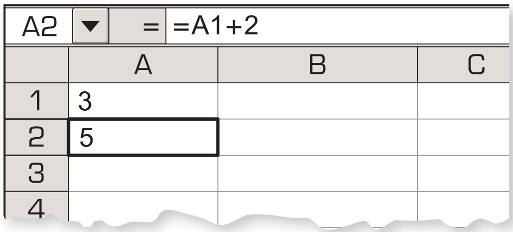
52. **MODELING WITH MATHEMATICS** The X-ray shows the lengths (in centimeters) of bones in a human hand.



a. Write a recursive rule for the lengths of the bones.

b. Measure the lengths of different sections of your hand. Can the lengths be represented by a recursively defined sequence? Explain.

53. USING TOOLS You can use a spreadsheet to generate the terms of a sequence.



A2 ▼ = =A1+2

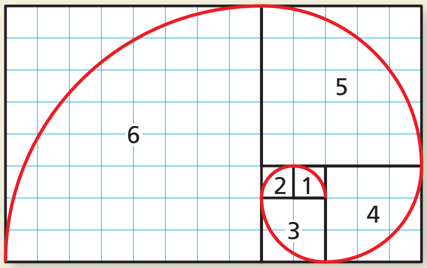
	A	B	C
1	3		
2	5		
3			
4			

a. To generate the terms of the sequence $a_1 = 3$, $a_n = a_{n-1} + 2$, enter the value of a_1, 3, into cell A1. Then enter "=A1+2" into cell A2, as shown. Use the *fill down* feature to generate the first 10 terms of the sequence.

b. Use a spreadsheet to generate the first 10 terms of the sequence $a_1 = 3$, $a_n = 4a_{n-1}$. (*Hint:* Enter "=4*A1" into cell A2.)

c. Use a spreadsheet to generate the first 10 terms of the sequence $a_1 = 4$, $a_2 = 7$, $a_n = a_{n-1} - a_{n-2}$. (*Hint:* Enter "=A2-A1" into cell A3.)

54. HOW DO YOU SEE IT? Consider Squares 1–6 in the diagram.



a. Write a sequence in which each term a_n is the side length of square n.

b. What is the name of this sequence? What is the next term of this sequence?

c. Use the term in part (b) to add another square to the diagram and extend the spiral.

55. REASONING Write the first 5 terms of the sequence $a_1 = 5$, $a_n = 3a_{n-1} + 4$. Determine whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

56. THOUGHT PROVOKING Describe the pattern for the numbers in Pascal's Triangle, shown below. Write a recursive rule that gives the *m*th number in the *n*th row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

57. REASONING The explicit rule $a_n = a_1 + (n - 1)d$ defines an arithmetic sequence.

a. Explain why $a_{n-1} = a_1 + [(n - 1) - 1]d$.

b. Justify each step in showing that a recursive equation for the sequence is $a_n = a_{n-1} + d$.

$$a_n = a_1 + (n - 1)d$$
$$= a_1 + [(n - 1) + 0]d$$
$$= a_1 + [(n - 1) - 1 + 1]d$$
$$= a_1 + [((n - 1) - 1) + 1]d$$
$$= a_1 + [(n - 1) - 1]d + d$$
$$= a_{n-1} + d$$

58. MAKING AN ARGUMENT Your friend claims that the sequence

$-5, 5, -5, 5, -5, \ldots$

cannot be represented by a recursive rule. Is your friend correct? Explain.

59. PROBLEM SOLVING Write a recursive rule for the sequence.

$3, 7, 15, 31, 63, \ldots$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

60. $5x + 12x$

61. $9 - 6y - 14$

62. $2d - 7 - 8d$

63. $3 - 3m + 11m$

Write a linear function *f* with the given values. *(Section 4.2)*

64. $f(2) = 6, f(-1) = -3$

65. $f(-2) = 0, f(6) = -4$

66. $f(-3) = 5, f(-1) = 5$

67. $f(3) = -1, f(-4) = -15$

6.5–6.7 What Did You Learn?

Core Vocabulary

exponential equation, *p. 326*
geometric sequence, *p. 332*
common ratio, *p. 332*
explicit rule, *p. 340*
recursive rule, *p. 340*

Core Concepts

Section 6.5

Property of Equality for Exponential Equations, *p. 326*
Solving Exponential Equations by Graphing, *p. 328*

Section 6.6

Geometric Sequence, *p. 332*
Equation for a Geometric Sequence, *p. 334*

Section 6.7

Recursive Equation for an Arithmetic Sequence, *p. 340*
Recursive Equation for a Geometric Sequence, *p. 340*

Mathematical Practices

1. How did you decide on an appropriate level of precision for your answer in Exercise 49 on page 330?
2. Explain how writing a function in Exercise 39 part (a) on page 337 created a shortcut for answering part (b).
3. How did you choose an appropriate tool in Exercise 52 part (b) on page 345?

Performance Task

The New Car

There is so much more to buying a new car than the purchase price. Interest rates, depreciation, and inflation are all factors. So, what is the real cost of your new car?

To explore the answers to this question and more, go to ***BigIdeasMath.com***.

6 Chapter Review

Dynamic Solutions available at *BigIdeasMath.com*

6.1 Properties of Exponents (pp. 291–298)

Simplify $\left(\frac{x}{4}\right)^{-4}$. Write your answer using only positive exponents.

$$\left(\frac{x}{4}\right)^{-4} = \frac{x^{-4}}{4^{-4}} \qquad \text{Power of a Quotient Property}$$

$$= \frac{4^4}{x^4} \qquad \text{Definition of negative exponent}$$

$$= \frac{256}{x^4} \qquad \text{Simplify.}$$

Simplify the expression. Write your answer using only positive exponents.

1. $y^3 \cdot y^{-5}$ **2.** $\frac{x^4}{x^7}$ **3.** $(x^0y^2)^3$ **4.** $\left(\frac{2x^2}{5y^4}\right)^{-2}$

6.2 Radicals and Rational Exponents (pp. 299–304)

Evaluate $512^{1/3}$.

$$512^{1/3} = \sqrt[3]{512} \qquad \text{Rewrite the expression in radical form.}$$

$$= \sqrt[3]{8 \cdot 8 \cdot 8} \qquad \text{Rewrite the expression showing factors.}$$

$$= 8 \qquad \text{Evaluate the cube root.}$$

Evaluate the expression.

5. $\sqrt[3]{8}$ **6.** $\sqrt[5]{-243}$ **7.** $625^{3/4}$ **8.** $(-25)^{1/2}$

6.3 Exponential Functions (pp. 305–312)

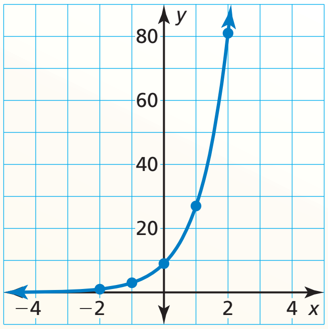
Graph $f(x) = 9(3)^x$.

Step 1 Make a table of values.

x	−2	−1	0	1	2
f(x)	1	3	9	27	81

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.



Graph the function. Describe the domain and range.

9. $f(x) = -4\left(\frac{1}{4}\right)^x$ **10.** $f(x) = 3^{x+2}$ **11.** $f(x) = 2^{x-4} - 3$

12. Write and graph an exponential function f represented by the table. Then compare the graph to the graph of $g(x) = \left(\frac{1}{2}\right)^x$.

x	0	1	2	3
y	2	1	0.5	0.25

6.4 Exponential Growth and Decay (pp. 313–322)

Rewrite the function $y = 10(0.65)^{t/8}$ to determine whether it represents *exponential growth* or *exponential decay*. Identify the percent rate of change.

$$
\begin{aligned}
y &= 10(0.65)^{t/8} && \text{Write the function.}\\
&= 10(0.65^{1/8})^t && \text{Power of a Power Property}\\
&\approx 10(0.95)^t && \text{Evaluate the power.}
\end{aligned}
$$

The function is of the form $y = a(1 - r)^t$, where $1 - r < 1$, so it represents exponential decay. Use the decay factor $1 - r$ to find the rate of decay.

$$
\begin{aligned}
1 - r &= 0.95 && \text{Write an equation.}\\
r &= 0.05 && \text{Solve for } r.
\end{aligned}
$$

▶ So, the function represents exponential decay, and the rate of decay is 5%.

Determine whether the table represents an *exponential growth function,* an *exponential decay function,* or *neither.* Explain.

13.

x	0	1	2	3
y	3	6	12	24

14.

x	1	2	3	4
y	162	108	72	48

Rewrite the function to determine whether it represents *exponential growth* or *exponential decay.* Identify the percent rate of change.

15. $f(t) = 4(1.25)^{t+3}$

16. $y = (1.06)^{8t}$

17. $f(t) = 6(0.84)^{t-4}$

18. You deposit \$750 in a savings account that earns 5% annual interest compounded quarterly. (a) Write a function that represents the balance after t years. (b) What is the balance of the account after 4 years?

19. The value of a TV is \$1500. Its value decreases by 14% each year. (a) Write a function that represents the value y (in dollars) of the TV after t years. (b) Find the approximate monthly percent decrease in value. (c) Graph the function from part (a). Use the graph to estimate the value of the TV after 3 years.

6.5 Solving Exponential Equations (pp. 325–330)

Solve $\frac{1}{9} = 3^{x+6}$.

$$
\begin{aligned}
\tfrac{1}{9} &= 3^{x+6} && \text{Write the equation.}\\
3^{-2} &= 3^{x+6} && \text{Rewrite } \tfrac{1}{9} \text{ as } 3^{-2}.\\
-2 &= x + 6 && \text{Equate the exponents.}\\
x &= -8 && \text{Solve for } x.
\end{aligned}
$$

Solve the equation.

20. $5^x = 5^{3x-2}$

21. $3^{x-2} = 1$

22. $-4 = 6^{4x-3}$

23. $\left(\frac{1}{3}\right)^{2x+3} = 5$

24. $\left(\frac{1}{16}\right)^{3x} = 64^{2(x+8)}$

25. $27^{2x+2} = 81^{x+4}$

6.6 Geometric Sequences (pp. 331–338)

Write the next three terms of the geometric sequence 2, 6, 18, 54, ...

Use a table to organize the terms and extend the sequence.

Position	1	2	3	4	5	6	7
Term	2	6	18	54	162	486	1458

× 3 × 3 × 3 × 3 × 3 × 3

Each term is 3 times the previous term. So, the common ratio is 3.

Multiply a term by 3 to find the next term.

▶ The next three terms are 162, 486, and 1458.

Decide whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. If the sequence is geometric, write the next three terms and graph the sequence.

26. 3, 12, 48, 192, . . .

27. 9, −18, 27, −36, . . .

28. 375, −75, 15, −3, . . .

Write an equation for the *n*th term of the geometric sequence. Then find a_9.

29. 1, 4, 16, 64, . . .

30. 5, −10, 20, −40, . . .

31. 486, 162, 54, 18, . . .

6.7 Recursively Defined Sequences (pp. 339–346)

Write a recursive rule for the sequence 5, 12, 19, 26, 33, ...

Use a table to organize the terms and find the pattern.

Position, *n*	1	2	3	4	5
Term, a_n	5	12	19	26	33

+ 7 + 7 + 7 + 7

The sequence is arithmetic, with first term $a_1 = 5$ and common difference $d = 7$.

$a_n = a_{n-1} + d$ Recursive equation for an arithmetic sequence

$a_n = a_{n-1} + 7$ Substitute 7 for *d*.

▶ So, a recursive rule for the sequence is $a_1 = 5$, $a_n = a_{n-1} + 7$.

Write the first six terms of the sequence. Then graph the sequence.

32. $a_1 = 4$, $a_n = a_{n-1} + 5$

33. $a_1 = -4$, $a_n = -3a_{n-1}$

34. $a_1 = 32$, $a_n = \frac{1}{4}a_{n-1}$

Write a recursive rule for the sequence.

35. 3, 8, 13, 18, 23, . . .

36. 3, 6, 12, 24, 48, . . .

37. 7, 6, 13, 19, 32, . . .

38. The first term of a sequence is 8. Each term of the sequence is 5 times the preceding term. Graph the first four terms of the sequence. Write a recursive rule and an explicit rule for the sequence.

6 Chapter Test

Evaluate the expression.

1. $-\sqrt[4]{16}$

2. $729^{1/6}$

3. $(-32)^{7/5}$

Simplify the expression. Write your answer using only positive exponents.

4. $z^{-2} \cdot z^{4}$

5. $\dfrac{b^{-5}}{a^{0}b^{-8}}$

6. $\left(\dfrac{2c^{4}}{5}\right)^{-3}$

Write and graph a function that represents the situation.

7. Your starting annual salary of $42,500 increases by 3% each year.

8. You deposit $500 in an account that earns 6.5% annual interest compounded yearly.

Write an explicit rule and a recursive rule for the sequence.

9.

n	1	2	3	4
a_n	−6	8	22	36

10.

n	1	2	3	4
a_n	400	100	25	6.25

Solve the equation. Check your solution.

11. $2^{x} = \dfrac{1}{128}$

12. $256^{x+2} = 16^{3x-1}$

13. Graph $f(x) = 2(6)^{x}$. Compare the graph to the graph of $g(x) = 6^{x}$. Describe the domain and range of f.

Use the equation to complete the statement "*a* ▭ *b*" with the symbol <, >, or =. Do not attempt to solve the equation.

14. $\dfrac{5^{a}}{5^{b}} = 5^{-3}$

15. $9^{a} \cdot 9^{-b} = 1$

16. The first two terms of a sequence are $a_1 = 3$ and $a_2 = -12$. Let a_3 be the third term when the sequence is arithmetic and let b_3 be the third term when the sequence is geometric. Find $a_3 - b_3$.

17. At sea level, Earth's atmosphere exerts a pressure of 1 atmosphere. Atmospheric pressure P (in atmospheres) decreases with altitude. It can be modeled by $P = (0.99988)^{a}$, where a is the altitude (in meters).

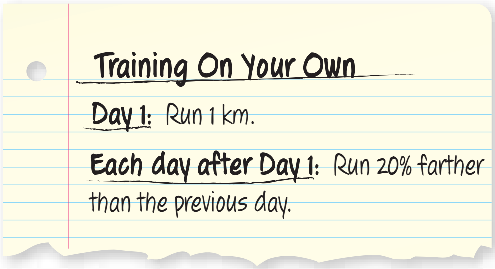
a. Identify the initial amount, decay factor, and decay rate.

b. Use a graphing calculator to graph the function. Use the graph to estimate the atmospheric pressure at an altitude of 5000 feet.

18. You follow the training schedule from your coach.

a. Write an explicit rule and a recursive rule for the geometric sequence.

b. On what day do you run approximately 3 kilometers?



Training On Your Own

Day 1: Run 1 km.

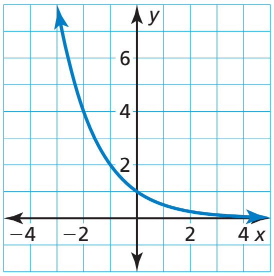
Each day after Day 1: Run 20% farther than the previous day.

6 Cumulative Assessment

1. Fill in the exponent of *x* with a number to simplify the expression.

$$\frac{x^{5/3} \cdot x^{-1} \cdot \sqrt[3]{x}}{x^{-2} \cdot x^{0}} = x^{\square}$$

2. The graph of the exponential function *f* is shown. Find $f(-7)$.



3. Student A claims he can form a linear system from the equations shown that has infinitely many solutions. Student B claims she can form a linear system from the equations shown that has one solution. Student C claims he can form a linear system from the equations shown that has no solution.

$3x + y = 12$	$3x + 2y = 12$	$6x + 2y = 6$
$3y + 9x = 36$	$2y - 6x = 12$	$9x - 3y = -18$

a. Select two equations to support Student A's claim.

b. Select two equations to support Student B's claim.

c. Select two equations to support Student C's claim.

4. Fill in the inequality with <, ≤, >, or ≥ so that the system of linear inequalities has no solution.

Inequality 1 $y - 2x \leq 4$

Inequality 2 $6x - 3y \;\square\; -12$

5. The second term of a sequence is 7. Each term of the sequence is 10 more than the preceding term. Fill in values to write a recursive rule and an explicit rule for the sequence.

$a_1 = \square, \; a_n = a_{n-1} + \square$

$a_n = \square\, n - \square$

6. A data set consists of the heights y (in feet) of a hot-air balloon t minutes after it begins its descent. An equation of the line of best fit is $y = 870 - 14.8t$. Which of the following is a correct interpretation of the line of best fit?

Ⓐ The initial height of the hot-air balloon is 870 feet. The slope has no meaning in this context.

Ⓑ The initial height of the hot-air balloon is 870 feet, and it descends 14.8 feet per minute.

Ⓒ The initial height of the hot-air balloon is 870 feet, and it ascends 14.8 feet per minute.

Ⓓ The hot-air balloon descends 14.8 feet per minute. The y-intercept has no meaning in this context.

7. Select all the functions whose x-value is an integer when $f(x) = 10$.

$f(x) = 3x - 2$	$f(x) = -2x + 4$	$f(x) = \frac{3}{2}x + 4$
$f(x) = -3x + 5$	$f(x) = \frac{1}{2}x - 6$	$f(x) = 4x + 14$

8. Place each function into one of the three categories. For exponential functions, state whether the function represents *exponential growth*, *exponential decay*, or *neither.*

Exponential	**Linear**	**Neither**
$f(x) = -2(8)^x$	$f(x) = 15 - x$	$f(x) = \frac{1}{2}(3)^x$
$f(x) = 6x^2 + 9$	$f(x) = 4(1.6)^{x/10}$	$f(x) = x(18 - x)$
$f(x) = 3\left(\frac{1}{6}\right)^x$	$f(x) = -3(4x + 1 - x)$	$f(x) = \sqrt[4]{16} + 2x$

9. How does the graph shown compare to the graph of $f(x) = 2^x$?